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具有4个相异指数随机变量的串联系统的条件样本间隔

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摘 要:基于具有4个独立不同指数分布元件的串联系统在某时刻失效时的元件剩余寿命,引入条件样本间隔概念,借助概率论与随机过程得到了其生存函数、期望和协方差等。然后,证明出了第一个条件样本间隔的生存函数关于元件的失效率是舒尔凸的。最后得到了第二个条件样本间隔在普通随机序意义下小于等于第三个条件样本间隔。所得结论将为设计最优系统和寿命检验提供理论依据。

关键词:条件样本间隔;串联系统;联合分布;随机序;指数分布
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The Conditional Sample Spacings of Series Systems with Four Divergence Components Random Variables

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Abstract: When a series system with four independent components distributed as exponential fails at some time, the conditional spacings are introduced. By probability and stochastic process, the survival functions, the expectation and covariance of conditional sample spacings are obtained. Then, the first spacing is proved to be Schur convex about the hazard rates of components lifetimes. In the end, a stochastic comparison between the second conditional spacing and the third one is investigated. The results obtained will provide theory basis for designing optimal system and life tests.

Keywords: conditional sample spacings; series systems; joint distribution; stochastic orders; exponential distributions

令$X_{i,n}$是独立且非负的随机变量$X_1,X_2,\cdots,X_n$中第i个顺序统计量。基于顺序统计量的样本间隔对应于生存领域系统中元件连续失效时刻之间的差:$D_{i,n}=\left(X_{i+1,n}-X_{i,n}\right),D_{i,n}^*=(n-i)\left(X_{i+1,n}-X_{i,n}\right),i\in\{1,2,\cdots,n-1\}$,其被分别称为样本间隔和标准样本间隔。过去许多学者研究了$D_{i,n}\left(D_{i,n}^*\right)$和$D_{i+1,n}\left(D_{i+1,n}^*\right)$之间的随机性质,例如,当元件寿命是任意连续分布且具有递减的失效率时,Pledger和Proschan[1]得到了$D_{i,n}^*\leqslant_{st}D_{i+1,n}^*$,$i\in\{1,2,\cdots,n-1\}$,Kochar和Korwar[2]将此结论从普通的随机序意义下加强到了似然比序意义下。关于更多它们的随机性质参考文献[3-4]。

指数分布及其对应的顺序统计量以及样本间隔在许多领域都发挥着重要作用,许多学者已经对独立不同分布的指数变量的样本间隔之间的随机性质进行了研究,例如Kochar和Rojo[5]等。Bapat和Beg[6]研究了样

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本中具有不同失效率参数的独立指数随机变量的顺序统计量的分布理论及间隔性质;最近Zhang等[7]指出,由n个元件构成的串联系统在某时刻失效时系统中还剩余$n-1$个没有失效的元件,这些没有失效的元件可以用来进行寿命试验,也可以用到其他系统中去。此外他们把这些没有失效元件的寿命看作是一组新的样本,这些新样本构成的样本间隔被称为条件样本间隔,即$D^t_{i,n-1}=\left(X_{i+1,n}-X_{i,n}|X_{1,n}=t\right)$,$D^{t,*}_{i,n-1}=(n-i)\left(X_{i+1,n}-X_{i,n}|X_{1,n}=t\right)$,$i\in\{1,2,\cdots,n-1\}$,$t>0$,在给定的元件寿命是独立同分布且为任意连续随机变量时,Zhang等[7]研究了条件样本间隔的生存函数及其随机性质。

对于元件寿命是独立不同分布的情况截至目前还没有研究者给予关注。因此本文主要研究基于具有不同失效率参数的独立指数随机变量的串联系统在某时刻失效时的条件样本间隔及其各种性质。

定义1 假设X和Y是2个元件的寿命,其分别具有分布函数$F(x)$和$G(x)$,以及对应的概率密度$f(x)$和$g(x)$,分别用$\bar{F}(x)=1-F(x)$和$\bar{G}(x)=1-G(x)$来表示其各自的生存函数。如果对于所有的x,有$\bar{F}(x)\leqslant\bar{G}(x)$,那么在普通随机序意义下,随机变量$X$比$Y$随机小,记作:$X\leqslant_{st}Y$,可参考文献[8]。

定义2 令$\boldsymbol{x}=(x_1,x_2,\cdots,x_n)$和$\boldsymbol{y}=(y_1,y_2,\cdots,y_n)$是两个$n$维向量,并用$x_{(1)}\leqslant x_{(2)}\leqslant\cdots\leqslant x_{(n)}$和$y_{(1)}\leqslant y_{(2)}\leqslant\cdots\leqslant y_{(n)}$分别表示$\boldsymbol{x}$和$\boldsymbol{y}$的分量按递增顺序的重新排列。对于$j\in\{1,2,\cdots,n-1\}$,如果$\sum_{i=1}^{j}y_{(i)}\geqslant\sum_{i=1}^{j}x_{(i)}$,并且$\sum_{i=1}^{n}y_{(i)}=\sum_{i=1}^{n}x_{(i)}$,则称$\boldsymbol{x}$在占优序意义下小于等于$\boldsymbol{y}$,记作$\boldsymbol{x}\stackrel{m}{\leqslant}\boldsymbol{y}$。一个定义在集合$\mathcal{A}\subset\mathbb{R}^n$上的实值函数$\phi$如果满足以下条件:

$$\boldsymbol{x}\stackrel{m}{\leqslant}\boldsymbol{y}\Rightarrow\phi(\boldsymbol{x})\leqslant(\geqslant)\phi(\mathbf{y}),$$

那么此函数在集合$\mathcal{A}$上被称为是舒尔凸(凹)的,详情请参考文献[9]。

在本文中,我们使用“递增”和“递减”分别表示“不减”和“不增”的含义。

1 主要结论及证明

令X_i是独立指数分布并且分别具有失效率参数$\lambda_i(i=1,2,3,4)$,设$s=\lambda_1+\lambda_2+\lambda_3+\lambda_4$,那么对于$i=1,2,3$和固定的$t>0$,令$D^t_{i,3}=\left(X_{i+1,4}-X_{i,4}|X_{1,4}=t\right)$和$D^{t,*}_{i,3}=(4-i)\left(X_{i+1,4}-X_{i,4}|X_{1,4}=t\right)$分别表示基于具有4个独立指数分布元件的串联系统的条件样本间隔和标准条件样本间隔。

对于固定时刻$t>0$,第一个条件样本间隔$\left(X_{2,4}-X_{1,4}|X_{1,4}=t\right)$的生存函数为

$$\begin{gathered}P\left(D^t_{1,3}>x\right)=P\left(X_{2,4}-X_{1,4}>x|X_{1,4}=t\right)=\\\sum_{i=1}^{4}P\left(X_{2,4}-X_{1,4}>x,X_{1,4}=X_i|X_{1,4}=t\right)=\\\sum_{i=1}^{4}P\left(X_{2,4}-X_{1,4}>x|X_{1,4}=X_i=t\right)P\left(X_{1,4}=X_i|X_{1,4}=t\right)=\\\sum_{i=1}^{4}e^{-\left(\lambda_j+\lambda_k+\lambda_m\right)x}\cdot\frac{\lambda_i}{s}=\\\sum_{i=1}^{4}\frac{\lambda_i}{s}e^{\left(\lambda_i-s\right)x},\end{gathered}$$

上述倒数第二个不等式成立是因为

$$\begin{gathered}P\left(X_{1,4}=X_i|X_{1,4}=t\right)=\\\frac{P\left(X_j>X_i,X_m>X_i,X_k>X_i,X_i=t\right)}{P\left(X_{1,4}=t\right)}=\frac{\lambda_i}{s},\end{gathered}$$

其中$\{i,j,k,m\}$是$\{1,2,3,4\}$的任意排列。

如上述第一个条件样本间隔的生存函数所示,可以得到

$$P\left(D_{1,3}^{t,*}>x\right)=\sum_{i=1}^{4}\frac{\lambda_i}{s}e^{\frac{\left(\lambda_i-s\right)x}{3}}。$$

定理1　对于固定的$t>0$，$D_{1,3}^{t}$的生存函数关于$\left(\lambda_1,\lambda_2,\lambda_3,\lambda_4\right)$是舒尔凸的。

证明　在每一个固定点$x>0$，对于$i=1,2,3,4$，显然$\lambda_i e^{\lambda_i x}$关于$\lambda_i$是凸的。因此由文献[10]可知，对于$x$，间隔$D_{1,3}^{t}$的生存函数$\sum_{i=1}^{4}\frac{\lambda_i}{s}e^{\left(\lambda_i-s\right)x}=\frac{e^{-sx}}{s}\sum_{i=1}^{4}\lambda_i e^{\lambda_i x}$关于失效率参数$\left(\lambda_1,\lambda_2,\lambda_3,\lambda_4\right)$是舒尔凸的，即结论成立。

对于固定时间t和任意的$x>0$，条件样本间隔$\left(X_{3,4}-X_{2,4}|X_{1,4}=t\right)$的生存函数为

$$\begin{gathered}P\left(D_{2,3}^{t}>x\right)=\\ P\left(X_{3,4}-X_{2,4}>x|X_{1,4}=t\right)=\\ \sum_{i=1}^{4}P\left(X_{3,4}-X_{2,4}>x,X_{1,4}=X_i|X_{1,4}=t\right)=\\ \sum_{i=1}^{4}P\left(X_{3,4}-X_{2,4}>x|X_{1,4}=X_i=t\right)P\left(X_{1,4}=X_i|X_{1,4}=t\right)。\end{gathered}\tag{1}$$

另外，在式(1)的最后一个等式中，对于固定的$t>0$和任意的$x>0$，条件概率$P\left(X_{3,4}-X_{2,4}>x|X_{1,4}=X_i=t\right)$的计算过程如下：

$$\begin{gathered}P\left(X_{3,4}-X_{2,4}>x|X_{1,4}=X_i=t\right)=\\ \sum_{j\neq i,j=1}^{4}P\left(X_{3,4}-X_{2,4}>x,X_{2,4}=X_j|X_{1,4}=X_i=t\right)=\\ \sum_{j\neq i,j=1}^{4}\int_t^{+\infty}P\left(X_{3,4}-y>x|X_{2,4}=X_j=y,X_{1,4}=X_i=t\right)P\left(X_{2,4}=X_j=y|X_{1,4}=X_i=t\right)\mathrm{d}y=\\ \sum_{j\neq i,j=1}^{4}\int_t^{+\infty}e^{-\left(\lambda_k+\lambda_m\right)x}\lambda_j e^{-\left(\lambda_j+\lambda_k+\lambda_m\right)\left(y-t\right)}\mathrm{d}y=\\ \sum_{j\neq i,j=1}^{4}\frac{\lambda_j}{\lambda_j+\lambda_k+\lambda_m}e^{-\left(\lambda_k+\lambda_m\right)x}。\end{gathered}\tag{2}$$

综上所述，由等式(2)可得第二个条件样本间隔的生存函数为

$$P\left(D_{2,3}^{t}>x\right)=P\left(X_{3,4}-X_{2,4}>x|X_{1,4}=t\right)=\frac{1}{s}\sum_{i=1}^{4}\sum_{j\neq i,j=1}^{4}\frac{\lambda_i\lambda_j}{\lambda_j+\lambda_k+\lambda_m}e^{-\left(\lambda_k+\lambda_m\right)x}。\tag{3}$$

注1　由式(3)可知，给定一个由4个独立但不同指数分布的元件构成的串联系统寿命为X_i的元件在t时刻失效时，第二个条件样本间隔的生存函数不依赖于其寿命。

如第二个条件样本间隔的生存函数，第二个标准条件样本间隔$D_{2,3}^{t,*}$的生存函数为

$$\begin{gathered}P\left(D_{2,3}^{t,*}>x\right)=\\ P\left(2\left(X_{3,4}-X_{2,4}|X_{1,4}=t\right)>x\right)=\\ \sum_{i=1}^{4}\sum_{j\neq i,j=1}^{4}\frac{\lambda_i}{s}\int_t^{+\infty}e^{-\frac{\left(\lambda_k+\lambda_m\right)x}{2}}\lambda_j e^{-\left(\lambda_j+\lambda_k+\lambda_m\right)\left(y-t\right)}\mathrm{d}y=\\ \frac{1}{s}\sum_{i=1}^{4}\sum_{j\neq i,j=1}^{4}\frac{\lambda_i\lambda_j}{\lambda_j+\lambda_k+\lambda_m}e^{-\frac{\left(\lambda_k+\lambda_m\right)x}{2}}。\end{gathered}$$

此外，对于固定的时间t和任意的$x>0$，第三个条件样本间隔$D_{3,3}^{t}$的生存函数为

$$\begin{gathered}P\left(D_{3,3}^{t}>x\right)=\\ P\left(X_{4,4}-X_{3,4}>x|X_{1,4}=t\right)=\\ \sum_{i=1}^{4}P\left(X_{4,4}-X_{3,4}>x,X_{1,4}=X_i|X_{1,4}=t\right)=\\ \sum_{i=1}^{4}P\left(X_{4,4}-X_{3,4}>x|X_{1,4}=X_i=t\right)P\left(X_{1,4}=X_i|X_{1,4}=t\right)。\end{gathered}$$

根据顺序统计量的马尔可夫性,我们计算上式中最后一个等式的条件概率$P\left(X_{4,4}-X_{3,4}>x|X_{1,4}=X_i=t\right)$如下：

$$
\begin{gathered}
P\left(X_{4,4}-X_{3,4}>x|X_{1,4}=X_i=t\right)= \\
\sum_{j\neq i,j=1}^{4} P\left(X_{4,4}-X_{3,4}>x,X_{2,4}=X_j|X_{1,4}=X_i=t\right)= \\
\sum_{j\neq i,j=1}^{4}\int_t^{+\infty} P\left(X_{4,4}-X_{3,4}>x,X_{2,4}=X_j=y|X_{1,4}=X_i=t\right)\mathrm{d}y= \\
\sum_{j\neq i,j=1}^{4}\int_t^{+\infty} P\left(X_{4,4}-X_{3,4}>x|X_{2,4}=X_j=y,X_{1,4}=X_i=t\right)P\left(X_{2,4}=X_j=y|X_{1,4}=X_i=t\right)\mathrm{d}y= \\
\sum_{j\neq i,j=1}^{4}\int_t^{+\infty}\frac{1}{\lambda_k+\lambda_m}\left(\lambda_k e^{-\lambda_m x}+\lambda_m e^{-\lambda_k x}\right)\lambda_j e^{-\left(\lambda_j+\lambda_k+\lambda_m\right)(y-t)}\mathrm{d}y= \\
\sum_{j\neq i,j=1}^{4}\frac{\lambda_j}{\lambda_j+\lambda_k+\lambda_m}\frac{\lambda_k e^{-\lambda_m x}+\lambda_m e^{-\lambda_k x}}{\lambda_k+\lambda_m}。
\end{gathered}
\tag{4}
$$

其中式(4)中第四个等式的成立基于以下事实：

$$
\begin{gathered}
P\left(X_{4,4}-X_{3,4}>x|X_{2,4}=X_j=y,X_{1,4}=X_i=t\right)= \\
P\left(X_{4,4}-X_{3,4}>x,X_{3,4}=X_k|X_{2,4}=X_j=y,X_{1,4}=X_i=t\right)+ \\
P\left(X_{4,4}-X_{3,4}>x,X_{3,4}=X_m|X_{2,4}=X_j=y,X_{1,4}=X_i=t\right)= \\
\int_y^{+\infty} P\left(X_{4,4}-X_{3,4}>x,X_{3,4}=X_k=z|X_{2,4}=X_j=y,X_{1,4}=X_i=t\right)\mathrm{d}z+ \\
\int_y^{+\infty} P\left(X_{4,4}-X_{3,4}>x,X_{3,4}=X_m=z|X_{2,4}=X_j=y,X_{1,4}=X_i=t\right)\mathrm{d}z= \\
\int_y^{+\infty} P\left(X_{4,4}-X_{3,4}>x|X_{3,4}=X_k=z,X_{2,4}=X_j=y,X_{1,4}=X_i=t\right)\times \\
P\left(X_{3,4}=X_k=z|X_{2,4}=X_j=y,X_{1,4}=X_i=t\right)\mathrm{d}z+ \\
\int_y^{+\infty} P\left(X_{4,4}-X_{3,4}>x|X_{3,4}=X_m=z,X_{2,4}=X_j=y,X_{1,4}=X_i=t\right)\times \\
P\left(X_{3,4}=X_m=z|X_{2,4}=X_j=y,X_{1,4}=X_i=t\right)\mathrm{d}z= \\
\int_y^{+\infty} e^{-\lambda_m x}\lambda_k e^{-\left(\lambda_k+\lambda_m\right)(z-y)}\mathrm{d}z+\int_y^{+\infty} e^{-\lambda_k x}\lambda_m e^{-\left(\lambda_k+\lambda_m\right)(z-y)}\mathrm{d}z= \\
\frac{1}{\lambda_k+\lambda_m}\left(\lambda_k e^{-\lambda_m x}+\lambda_m e^{-\lambda_k x}\right),
\end{gathered}
$$

并且式(4)中第三个等式的条件概率$P\left(X_{2,4}=X_j=y|X_{1,4}=X_i=t\right)$的计算如下：

$$
\begin{gathered}
P\left(X_{2,4}=X_j=y|X_{1,4}=X_i=t\right)= \\
\frac{P\left(X_{2,4}=X_j=y,X_{1,4}=X_i=t\right)}{P\left(X_{1,4}=X_i=t\right)}= \\
\frac{\lambda_i\lambda_j e^{-\left(\lambda_j+\lambda_k+\lambda_m\right)y}e^{-\lambda_i t}}{\lambda_i e^{-\left(\lambda_i+\lambda_j+\lambda_k+\lambda_m\right)t}}= \\
\lambda_j e^{-\left(\lambda_j+\lambda_k+\lambda_m\right)(y-t)}。
\end{gathered}
$$

综上所述,由等式(4)可知第三个条件样本间隔的生存函数如下：

$$
P\left(D_{3,3}^t>x\right)=P\left(X_{4,4}-X_{3,4}>x|X_{1,4}=t\right)=\frac{1}{s}\sum_{i=1}^{4}\sum_{i\neq j,j=1}^{4}\frac{\lambda_i\lambda_j}{\lambda_j+\lambda_k+\lambda_m}\frac{\lambda_k e^{-\lambda_m x}+\lambda_m e^{-\lambda_k x}}{\lambda_k+\lambda_m}。
$$

对于任意的$x>0$、$y>0$、$z>0$和固定的时间$t>0$,前三个间隔的联合生存函数为

$$
\begin{gathered}
P\left(D_{1,3}^{t}>x,D_{2,3}^{t}>y,D_{3,3}^{t}>z\right)= \\
P\left(X_{2,4}-X_{1,4}>x,X_{3,4}-X_{2,4}>y,X_{4,4}-X_{3,4}>z|X_{1,4}=t\right)= \\
\sum_{i=1}^{4}P\left(X_{2,4}-t>x,X_{3,4}-X_{2,4}>y,X_{4,4}-X_{3,4}>z,X_{1,4}=X_i|X_{1,4}=t\right)= \\
\sum_{i=1}^{4}P\left(X_{2,4}-t>x,X_{3,4}-X_{2,4}>y,X_{4,4}-X_{3,4}>z|X_{1,4}=X_i=t\right)\times \\
P\left(X_{1,4}=X_i|X_{1,4}=t\right)= \\
\sum_{i=1}^{4}\sum_{j\neq i,j=1}^{4}P\left(X_{2,4}-t>x,X_{3,4}-X_{2,4}>y,X_{4,4}-X_{3,4}>z,X_{2,4}=X_j|X_{1,4}=X_i=t\right)\times \\
P\left(X_{1,4}=X_i|X_{1,4}=t\right)= \\
\sum_{i=1}^{4}\sum_{j\neq i,j=1}^{4}\int_{t}^{+\infty}P\left(X_{2,4}-t>x,X_{3,4}-X_{2,4}>y,X_{4,4}-X_{3,4}>z,X_{2,4}=X_j=h|X_{1,4}=X_i=t\right)\mathrm{d}h\times \\
P\left(X_{1,4}=X_i|X_{1,4}=t\right)。
\end{gathered}
\tag{5}
$$

我们注意到，对于固定的$t>0$，式(5)的最后一个等式中的积分计算如下：

$$
\begin{gathered}
\int_{t}^{+\infty}P\left(X_{2,4}-t>x,X_{3,4}-X_{2,4}>y,X_{4,4}-X_{3,4}>z,X_{2,4}=X_j=h|X_{1,4}=X_i=t\right)\mathrm{d}h= \\
\int_{t+x}^{+\infty}P\left(X_{3,4}-h>y,X_{4,4}-X_{3,4}>z|X_{2,4}=X_j=h,X_{1,4}=X_i=t\right)\times \\
P\left(X_{2,4}=X_j=h|X_{1,4}=X_i=t\right)\mathrm{d}h= \\
\int_{t+x}^{+\infty}\sum_{m\neq i\neq j,m=1}^{4}P\left(X_{3,4}-h>y,X_{4,4}-X_{3,4}>z,X_{3,4}=X_m|X_{2,4}=X_j=h,X_{1,4}=X_i=t\right)\times \\
P\left(X_{2,4}=X_j=h|X_{1,4}=X_i=t\right)\mathrm{d}h= \\
\int_{t+x}^{+\infty}\sum_{m\neq i\neq j,m=1}^{4}P\left(X_{3,4}-h>y,X_{4,4}-X_{3,4}>z|X_{3,4}=X_m,X_{2,4}=X_j=h,X_{1,4}=X_i=t\right)\times \\
P\left(X_{3,4}=X_m|X_{2,4}=X_j=h,X_{1,4}=X_i=t\right)\times P\left(X_{2,4}=X_j=h|X_{1,4}=X_i=t\right)\mathrm{d}h= \\
\int_{t+x}^{+\infty}\frac{e^{-\left(\lambda_k+\lambda_m\right)y}}{\lambda_k+\lambda_m}\left(\lambda_k e^{-\lambda_m z}+\lambda_m e^{-\lambda_k z}\right)\lambda_j e^{-\left(\lambda_j+\lambda_k+\lambda_m\right)(h-t)}\mathrm{d}h= \\
\frac{\lambda_j e^{-\lambda_j x-\left(\lambda_k+\lambda_m\right)(x+y)}}{\left(\lambda_k+\lambda_m\right)\left(\lambda_j+\lambda_k+\lambda_m\right)}\left(\lambda_k e^{-\lambda_m z}+\lambda_m e^{-\lambda_k z}\right)。
\end{gathered}
\tag{6}
$$

其中式(6)中的第四个等式成立基于下述事实：

$$
\begin{gathered}
P\left(X_{3,4}-h>y,X_{4,4}-X_{3,4}>z|X_{3,4}=X_m,X_{2,4}=X_j=h,X_{1,4}=X_i=t\right)\times \\
P\left(X_{3,4}=X_m|X_{2,4}=X_j=h,X_{1,4}=X_i=t\right)= \\
\int_{h}^{+\infty}P\left(X_{3,4}-h>y,X_{4,4}-X_{3,4}>z|X_{3,4}=X_m=u,X_{2,4}=X_j=h,X_{1,4}=X_i=t\right)\times \\
P\left(X_{3,4}=X_m=u|X_{2,4}=X_j=h,X_{1,4}=X_i=t\right)\mathrm{d}u= \\
\int_{h+y}^{+\infty}P\left(X_{4,4}-u>z|X_{3,4}=X_m=u,X_{2,4}=X_j=h,X_{1,4}=X_i=t\right)\times \\
P\left(X_{3,4}=X_m=u|X_{2,4}=X_j=h,X_{1,4}=X_i=t\right)\mathrm{d}u= \\
\int_{h+y}^{+\infty}e^{-\lambda_k z}\lambda_m e^{-\left(\lambda_k+\lambda_m\right)(u-h)}\mathrm{d}u= \\
\frac{\lambda_m}{\lambda_k+\lambda_m}e^{-\lambda_k z}e^{-\left(\lambda_k+\lambda_m\right)y}。
\end{gathered}
$$

所以，对于固定的$t>0$，对上式求和如下：

$$\sum_{m \neq i \neq j, m=1}^{4} P\left(X_{3,4}-h>y, X_{4,4}-X_{3,4}>z \mid X_{3,4}=X_m, X_{2,4}=X_j=h, X_{1,4}=X_i=t\right) \times$$
$$P\left(X_{3,4}=X_m \mid X_{2,4}=X_j=h, X_{1,4}=X_i=t\right)=$$
$$\frac{\lambda_m}{\lambda_k+\lambda_m} e^{-\lambda_k z} e^{-\left(\lambda_k+\lambda_m\right) y}+\frac{\lambda_k}{\lambda_k+\lambda_m} e^{-\lambda_m z} e^{-\left(\lambda_k+\lambda_m\right) y}=$$
$$\frac{e^{-\left(\lambda_k+\lambda_m\right) y}}{\lambda_k+\lambda_m}\left(\lambda_m e^{-\lambda_k z}+\lambda_k e^{-\lambda_m z}\right)。$$

因此,由式(5)和式(6)可知,对于任意的$x>0, y>0, z>0$,

$$P\left(D_{1,3}^t>x, D_{2,3}^t>y, D_{3,3}^t>z\right)=$$
$$\sum_{i=1}^{4} \sum_{j \neq i, j=1}^{4} \frac{\lambda_i}{\lambda_1+\lambda_2+\lambda_3+\lambda_4} \frac{\lambda_j e^{-\lambda_j x-\left(\lambda_k+\lambda_m\right)(x+y)}}{\left(\lambda_k+\lambda_m\right)\left(\lambda_j+\lambda_k+\lambda_m\right)}\left(\lambda_k e^{-\lambda_m z}+\lambda_m e^{-\lambda_k z}\right)=$$
$$\sum_{i=1}^{4} \sum_{j \neq i, j=1}^{4} \frac{\lambda_i \lambda_j e^{-\lambda_j x-\left(\lambda_k+\lambda_m\right)(x+y)}}{\left(\lambda_1+\lambda_2+\lambda_3+\lambda_4\right)\left(\lambda_k+\lambda_m\right)\left(\lambda_j+\lambda_k+\lambda_m\right)}\left(\lambda_k e^{-\lambda_m z}+\lambda_m e^{-\lambda_k z}\right)=$$
$$\sum_{i=1}^{4} \frac{\lambda_i e^{\left(\lambda_i-s\right) x}}{s} \sum_{j \neq i, j=1}^{4} \frac{\lambda_j e^{-\left(\lambda_k+\lambda_m\right) y}}{\left(\lambda_k+\lambda_m\right)\left(\lambda_j+\lambda_k+\lambda_m\right)}\left(\lambda_k e^{-\lambda_m z}+\lambda_m e^{-\lambda_k z}\right)。$$

下面我们来探究$D_{2,3}^t=\left(X_{3,4}-X_{2,4} \mid X_{1,4}=t\right)$和$D_{3,3}^t=\left(X_{4,4}-X_{3,4} \mid X_{1,4}=t\right)$的联合分布。注意到,对于任意的$x>0$和$y>0$,

$$P\left(D_{2,3}^t>x, D_{3,3}^t>y\right)=$$
$$P\left(X_{3,4}-X_{2,4}>x, X_{4,4}-X_{3,4}>y \mid X_{1,4}=t\right)=$$
$$\sum_{i=1}^{4} P\left(X_{3,4}-X_{2,4}>x, X_{4,4}-X_{3,4}>y, X_{1,4}=X_i \mid X_{1,4}=t\right)=$$
$$\sum_{i=1}^{4} P\left(X_{3,4}-X_{2,4}>x, X_{4,4}-X_{3,4}>y \mid X_{1,4}=X_i=t\right) \times$$
$$P\left(X_{1,4}=X_i \mid X_{1,4}=t\right)=$$
$$\sum_{i=1}^{4} \sum_{j \neq i, j=1}^{4} P\left(X_{3,4}-X_{2,4}>x, X_{4,4}-X_{3,4}>y, X_{2,4}=X_j \mid X_{1,4}=X_i=t\right) \times$$
$$P\left(X_{1,4}=X_i \mid X_{1,4}=t\right)=$$
$$\sum_{i=1}^{4} \sum_{j \neq i, j=1}^{4} \int_t^{+\infty} P\left(X_{3,4}-X_{2,4}>x, X_{4,4}-X_{3,4}>y, X_{2,4}=X_j=z \mid X_{1,4}=X_i=t\right) \mathrm{d} z \times$$
$$P\left(X_{1,4}=X_i \mid X_{1,4}=t\right), \tag{7}$$

其中最后一个等式中的积分计算如下:

$$\int_t^{+\infty} P\left(X_{3,4}-X_{2,4}>x, X_{4,4}-X_{3,4}>y, X_{2,4}=X_j=z \mid X_{1,4}=X_i=t\right) \mathrm{d} z=$$
$$\int_t^{+\infty} P\left(X_{3,4}-X_{2,4}>x, X_{4,4}-X_{3,4}>y \mid X_{2,4}=X_j=z, X_{1,4}=X_i=t\right) \times$$
$$P\left(X_{2,4}=X_j=z \mid X_{1,4}=X_i=t\right) \mathrm{d} z=$$
$$\int_t^{+\infty} \sum_{m \neq i \neq j, m=1}^{4} P\left(X_{3,4}-X_{2,4}>x, X_{4,4}-X_{3,4}>y, X_{3,4}=X_m \mid X_{2,4}=X_j=z, X_{1,4}=X_i=t\right) \times$$
$$P\left(X_{2,4}=X_j=z \mid X_{1,4}=X_i=t\right) \mathrm{d} z=$$
$$\int_t^{+\infty} \frac{e^{-\left(\lambda_k+\lambda_m\right) x}}{\lambda_k+\lambda_m}\left(\lambda_k e^{-\lambda_m y}+\lambda_m e^{-\lambda_k y}\right) \lambda_j e^{-\left(\lambda_j+\lambda_k+\lambda_m\right)(z-t)} \mathrm{d} z=$$
$$\frac{\lambda_j e^{-\left(\lambda_k+\lambda_m\right) x}}{\left(\lambda_k+\lambda_m\right)\left(\lambda_j+\lambda_k+\lambda_m\right)}\left(\lambda_k e^{-\lambda_m y}+\lambda_m e^{-\lambda_k y}\right), \tag{8}$$

上式中第三个等式的成立基于下述式(9)和式(10)的计算结果：

$$
\begin{aligned}
&P\left(X_{3,4}-X_{2,4}>x,X_{4,4}-X_{3,4}>y,X_{3,4}=X_m|X_{2,4}=X_j=z,X_{1,4}=X_i=t\right)=\\
&\int_z^{+\infty}\left(X_{3,4}-X_{2,4}>x,X_{4,4}-X_{3,4}>y,X_{3,4}=X_m=h|X_{2,4}=X_j=z,X_{1,4}=X_i=t\right)\mathrm{d}h=\\
&\int_{z+x}^{+\infty}P\left(X_{4,4}-h>y|X_{3,4}=X_m=h,X_{2,4}=X_j=z,X_{1,4}=X_i=t\right)\times\\
&P\left(X_{3,4}=X_m=h|X_{2,4}=X_j=z,X_{1,4}=X_i=t\right)\mathrm{d}h=\\
&\int_{z+x}^{+\infty}e^{-\lambda_k y}\lambda_m e^{-(\lambda_k+\lambda_m)(h-z)}\mathrm{d}h=\\
&\frac{\lambda_m}{\lambda_k+\lambda_m}e^{-\lambda_k y}e^{-(\lambda_k+\lambda_m)x},
\end{aligned}
\tag{9}
$$

因此，

$$
\begin{aligned}
&\sum_{m\neq i\neq j,m=1}^{4}P\left(X_{3,4}-X_{2,4}>x,X_{4,4}-X_{3,4}>y,X_{3,4}=X_m|X_{2,4}=X_j=z,X_{1,4}=X_i=t\right)=\\
&\frac{\lambda_m}{\lambda_k+\lambda_m}e^{-\lambda_k y}e^{-(\lambda_k+\lambda_m)x}+\frac{\lambda_k}{\lambda_k+\lambda_m}e^{-\lambda_m y}e^{-(\lambda_k+\lambda_m)x}=\\
&\frac{e^{-(\lambda_k+\lambda_m)x}}{\lambda_k+\lambda_m}\left(\lambda_m e^{-\lambda_k y}+\lambda_k e^{-\lambda_m y}\right)。
\end{aligned}
\tag{10}
$$

又由于对于固定的 $t>0$ 且 $i\neq j$ 时，可以得到式(8)中第三个等式的条件概率 $P\left(X_{2,4}=X_j=z|X_{1,4}=X_i=t\right)$ 的计算过程如下：

$$
\begin{aligned}
&P\left(X_{2,4}=X_j=z|X_{1,4}=X_i=t\right)=\\
&\frac{P\left(X_{2,4}=X_j=z,X_{1,4}=X_i=t\right)}{P\left(X_{1,4}=X_i=t\right)}=\\
&\frac{P\left(X_m>z,X_k>z,X_j=z,X_i=t\right)}{P\left(X_m>t,X_j>t,X_k>t,X_i=t\right)}=\\
&\frac{e^{-\lambda_m z}e^{-\lambda_k z}\lambda_j e^{-\lambda_j z}\lambda_i e^{-\lambda_i z}}{e^{-\lambda_m t}e^{-\lambda_j t}e^{-\lambda_k t}\lambda_i e^{-\lambda_i z}}=\\
&\lambda_j e^{-(\lambda_j+\lambda_k+\lambda_m)(z-t)}。
\end{aligned}
\tag{11}
$$

综上所述，由式(7)、式(8)和式(11)可以得到

$$
\begin{aligned}
&P\left(D_{2,3}^t>y,D_{3,3}^t>z\right)=\\
&\sum_{i=1}^{4}\sum_{j\neq i,j=1}^{4}\frac{\lambda_i}{s}\frac{\lambda_j e^{-(\lambda_k+\lambda_m)y}}{\left(\lambda_k+\lambda_m\right)\left(\lambda_j+\lambda_k+\lambda_m\right)}\left(\lambda_k e^{-\lambda_m z}+\lambda_m e^{-\lambda_k z}\right)=\\
&\frac{1}{s}\sum_{i=1}^{4}\sum_{j\neq i,j=1}^{4}\frac{\lambda_i\lambda_j e^{-(\lambda_k+\lambda_m)y}}{\left(\lambda_k+\lambda_m\right)\left(\lambda_j+\lambda_k+\lambda_m\right)}\left(\lambda_k e^{-\lambda_m z}+\lambda_m e^{-\lambda_k z}\right)。
\end{aligned}
$$

下面求条件样本间隔的期望和方差：

$$
E\left(D_{2,3}^t\right)=\frac{1}{s}\sum_{i=1}^{4}\sum_{j\neq i,j=1}^{4}\frac{\lambda_i\lambda_j}{\left(\lambda_k+\lambda_m\right)\left(\lambda_j+\lambda_k+\lambda_m\right)},
$$

$$
E\left(D_{3,3}^t\right)=\frac{1}{s}\sum_{i=1}^{4}\sum_{j\neq i,j=1}^{4}\frac{\lambda_i\lambda_j\left(\lambda_k^2+\lambda_m^2\right)}{\lambda_k\lambda_m\left(\lambda_k+\lambda_m\right)\left(\lambda_j+\lambda_k+\lambda_m\right)},
$$

$$
E\left(D_{2,3}^tD_{3,3}^t\right)=\sum_{i=1}^{4}\sum_{j\neq i,j=1}^{4}\frac{\lambda_i\lambda_j\left(\lambda_k^2+\lambda_m^2\right)}{\lambda_k\lambda_m\left(\lambda_1+\lambda_2+\lambda_3+\lambda_4\right)\left(\lambda_k+\lambda_m\right)^2\left(\lambda_j+\lambda_k+\lambda_m\right)}。
$$

因此,第二个条件样本间隔$D_{2,3}^t$和第三个条件样本间隔$D_{3,3}^t$的协方差为

$$\mathrm{Cov}\left(D_{2,3}^t,D_{3,3}^t\right)=$$
$$E\left(D_{2,3}^t D_{3,3}^t\right)-E\left(D_{2,3}^t\right)\left(D_{3,3}^t\right)=$$
$$\frac{1}{s}\sum_{i=1}^{4}\sum_{j\neq i,j=1}^{4}\frac{\lambda_i\lambda_j\left(\lambda_k{}^2+\lambda_m{}^2\right)}{\lambda_k\lambda_m\left(\lambda_1+\lambda_2+\lambda_3+\lambda_4\right)\left(\lambda_k+\lambda_m\right)^2\left(\lambda_j+\lambda_k+\lambda_m\right)}-$$
$$\left[\sum_{i=1}^{4}\sum_{j\neq i,j=1}^{4}\frac{\lambda_i\lambda_j}{\left(\lambda_k+\lambda_m\right)\left(\lambda_j+\lambda_k+\lambda_m\right)}\right]\times\left[\sum_{i=1}^{4}\sum_{j\neq i,j=1}^{4}\frac{\lambda_i\lambda_j\left(\lambda_k{}^2+\lambda_m{}^2\right)}{\lambda_k\lambda_m\left(\lambda_k+\lambda_m\right)\left(\lambda_j+\lambda_k+\lambda_m\right)}\right]。$$

定理2 对于所有的$x\geqslant 0$,固定时刻$t>0$,$D_{3,3}^t\geqslant_{st}D_{2,3}^t$。

证明 由上述结论可知,

$$P\left(D_{3,3}^t>x\right)-P\left(D_{2,3}^t>x\right)=$$
$$\frac{1}{s}\sum_{i=1}^{4}\sum_{j\neq i,j=1}^{4}\frac{\lambda_i\lambda_j}{\lambda_j+\lambda_k+\lambda_m}\left(\frac{\lambda_k e^{-\lambda_m x}+\lambda_m e^{-\lambda_k x}}{\lambda_k+\lambda_m}-e^{-\left(\lambda_k+\lambda_m\right)x}\right)=$$
$$\frac{1}{s}\sum_{i=1}^{4}\sum_{j\neq i,j=1}^{4}\frac{\lambda_i\lambda_j}{\lambda_j+\lambda_k+\lambda_m}e^{-\left(\lambda_k+\lambda_m\right)x}\left(\frac{\lambda_k e^{-\lambda_k x}+\lambda_m e^{-\lambda_m x}}{\lambda_k+\lambda_m}-1\right)。$$

令函数$f(x)=\dfrac{\lambda_k e^{\lambda_k x}+\lambda_m e^{\lambda_m x}}{\lambda_k+\lambda_m}$,显然,对于所有的$x\geqslant 0$,$f(x)$是递增的,并且$f(0)=1$。对于所有的$x\geqslant 0$,$f(x)\geqslant f(0)$,因此和式里面的部分是非负的,即得结论。

参考文献:

[1] PLEDGER G,PROSCHAN F.Comparisons of order statistics and of spacings from heterogeneous distributions[J]. Optimizing Methods in Statistics,1971,89-113.

[2] KOCHAR S C,KORWAR R. Stochastic orders for spacings of heterogeneous exponential random variables[J]. Journal of Multivariate Analysis,1996,57(1):69-83.

[3] SHAKED M,TONG Y L. Stochastic ordering of spacings from dependent random variables[M]//Institute of Mathematical Statistics Lecture Notes-Monograph Series.Hayward,CA:Institute of Mathematical Statistics,1984:141-149.

[4] NAPPO G,SPIZZICHINO F. Ordering properties of the TTT-plot of lifetimes with Schur joint densities[J]. Statistics & Probability Letters,1998,39(3):195-203.

[5] KOCHAR S,ROJO J. Some new results on stochastic comparisons of spacings from heterogeneous exponential distributions[J]. Journal of Multivariate Analysis,1996,59(2):272-281.

[6] BAPAT R,BEG M I. Order statistics for nonidentically distributed variables and permanents[J]. Sankhyā Ser. A,1989,79-83

[7] ZHANG Z C,YANG L Y,YANG Y H. The conditional spacings and their stochastic properties[J]. Statistics & Probability Letters,2022,186:109466.

[8] SHAKED M,SHANTHIKUMAR J G. Stochastic orders[M]. New York:Springer,2007.

[9] ROBERTSON T,WRIGHT F T. On measuring the conformity of a parameter set to a trend, with applications[J]. The Annals of Statistics,1982,10(4):1234-1245.

[10] MARSHALL A W,OLKIN I,ARNOLD B C. Multivariate majorization[M]//Inequalities: theory of majorization and its applications. New York:Springer,2010:611-633.

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