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各向异性电介质静电场泊松方程的变分问题

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摘 要:根据对称正定算子方程的变分原理,将各向异性电介质静电场泊松方程第一和第二边值问题转化为与之等价的变分问题。发现泊松方程实质上是与之等价的变分问题中所构造泛函取得极值时所满足的奥斯特罗格拉茨基方程。由泛函的核函数定义了各向异性电介质静电场的能量密度,并由该概念出发推导出能量密度泛函。富有物理意义的是各向异性电介质静电场的泊松方程是能量密度泛函取得极值的必要条件,泊松方程的电势解是能量密度泛函取得极值的极值函数。

关键词:各向异性电介质;泊松方程;对称正定算子;变分问题;能量密度泛函
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Variational Problem of Poisson Equation in Electrostatic Field of Anisotropic Dielectric

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Abstract: The first and the second boundary value problem for Poisson equation in electrostatic of anisotropic dielectric was transformed into an equivalent variational problem, according to the variational principle of symmetric positive operator equation. It is founded that Poisson equation is essentially the Ostrongradski equation that is satisfied when the extremum of the functional constructed for the equivalent variational problem is obtained. The energy density in electrostatic of anisotropic dielectric is defined by kernel function of the functional in the variational problem, and then the energy density functional is derived from the concept. It has great physical signification that the Poisson equation in electrostatic of anisotropic dielectric is the necessary condition for the energy density functional to obtain the extreme value and the potential solution of Poisson equation is the extreme function of the energy density functional to obtain the extreme value.

Keywords: anisotropic dielectric; Poisson equation; symmetric positive operator; variational problem; energy density functional

各向异性电介质(限于仅有3个正交主轴方向的电介质)静电场的泊松方程是研究各向异性电介质静电场属性的微分方程。已有相关的一些文献研究解泊松方程的方法(如分离变量法、积分变换法)并结合具体的物理模型对各向异性电介质静电场属性进行了研究或将泊松方程作新应用[1-9]。本文不从解泊松方程的具体方法技巧入手,而是根据对称正定算子方程的变分原理[10]去构造泛函,从泛函的角度去理解各向异性电介质静电场的泊松方程的物理意义。所构造的泛函取得极值的变分问题与各向异性电介质静电场泊松方程的第一和第二边值问题等价,称为各向异性电介质静电场泊松方程的变分问题,可知在构造相应的泛函并应用变分原理时,就自然得到了泊松方程满足第一类或第二类齐次边界条件。求解所构造的泛函取得极

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值时,发现静电势解所满足的奥斯特罗格莱茨基方程(奥氏方程)实质上就是泊松方程。

1 各向异性电介质泊松方程的两种具体形式

在各向异性电介质主轴坐标系$O-x_1x_2x_3$中,泊松方程的具体形式为[1]

$$\varepsilon_{11}\frac{\partial^2 u\left(x_1,x_2,x_3\right)}{\partial {x_1}^2}+\varepsilon_{22}\frac{\partial^2 u\left(x_1,x_2,x_3\right)}{\partial {x_2}^2}+\varepsilon_{33}\frac{\partial^2 u\left(x_1,x_2,x_3\right)}{\partial {x_3}^2}=-\rho\left(x_1,x_2,x_3\right), \tag{1}$$

式中,ε_{11}、ε_{22}、ε_{33}分别表示沿着各向异性电介质主轴x_1、x_2、x_3方向的介电常数,$u\left(x_1,x_2,x_3\right)$是静电势,$\rho\left(x_1,x_2,x_3\right)$是电荷体密度。若$\rho\left(x_1,x_2,x_3\right)=0$,则式(1)退化为在各向异性电介质主轴坐标系$O-x_1x_2x_3$中拉普拉斯方程的具体形式。

进行坐标变换,令$x_1=\sqrt{\varepsilon_{11}}\,\xi_1$,$x_2=\sqrt{\varepsilon_{22}}\,\xi_2$,$x_3=\sqrt{\varepsilon_{33}}\,\xi_3$,有$\dfrac{\partial u}{\partial x_1}=\dfrac{\partial u}{\partial \xi_1}\dfrac{\partial \xi_1}{\partial x_1}=\dfrac{1}{\sqrt{\varepsilon_{11}}}\dfrac{\partial u}{\partial \xi_1}$,$\dfrac{\partial^2 u}{\partial {x_1}^2}=\dfrac{\partial}{\partial x_1}\left(\dfrac{\partial u}{\partial x_1}\right)=\dfrac{\partial}{\partial x_1}\left(\dfrac{1}{\sqrt{\varepsilon_{11}}}\dfrac{\partial u}{\partial \xi_1}\right)=\dfrac{1}{\sqrt{\varepsilon_{11}}}\dfrac{\partial}{\partial \xi_1}\left(\dfrac{\partial u}{\partial \xi_1}\right)\dfrac{\partial \xi_1}{\partial x_1}=\dfrac{1}{\varepsilon_{11}}\dfrac{\partial^2 u}{\partial {\xi_1}^2}$,同理求得$\dfrac{\partial^2 u}{\partial {x_2}^2}=\dfrac{\partial^2 u}{\partial {\xi_2}^2}$,$\dfrac{\partial^2 u}{\partial {x_3}^2}=\dfrac{\partial^2 u}{\partial {\xi_3}^2}$。将这些关系代入式(1)中,得到在电各向异性坐标系$O-\xi_1\xi_2\xi_3$中泊松方程的具体形式为

$$\frac{\partial^2 u_\xi\left(\xi_1,\xi_2,\xi_3\right)}{\partial {\xi_1}^2}+\frac{\partial^2 u_\xi\left(\xi_1,\xi_2,\xi_3\right)}{\partial {\xi_2}^2}+\frac{\partial^2 u_\xi\left(\xi_1,\xi_2,\xi_3\right)}{\partial {\xi_3}^2}=-\rho_\xi\left(\xi_1,\xi_2,\xi_3\right), \tag{2}$$

式中,$u_\xi\left(\xi_1,\xi_2,\xi_3\right)$是静电势,$\rho_\xi\left(\xi_1,\xi_2,\xi_3\right)$是电荷体密度。若$\rho_\xi\left(\xi_1,\xi_2,\xi_3\right)=0$,则式(2)退化为在电各向异性坐标系$O-\xi_1\xi_2\xi_3$中拉普拉斯方程的具体形式。文中下标或上标是"$\xi$"的量表示在电各向异性坐标系中的物理量或数学量。

2 各向异性电介质静电场泊松方程第一和第二边值问题的变分问题

在电各向异性坐标系$O-\xi_1\xi_2\xi_3$中定义算子$T=-\nabla_\xi\cdot\nabla_\xi=-\triangle_\xi=-\left(\dfrac{\partial^2}{\partial {\xi_1}^2}+\dfrac{\partial^2}{\partial {\xi_2}^2}+\dfrac{\partial^2}{\partial {\xi_3}^2}\right)$,于是泊松方程式(2)的算子方程的形式为

$$Tu_\xi\left(\xi_1,\xi_2,\xi_3\right)=\rho_\xi\left(\xi_1,\xi_2,\xi_3\right)。 \tag{3}$$

为了应用对称正定算子的变分原理构造与式(3)相对应的泛函,现先证明算子T是对称算子。在电各向异性坐标系$O-\xi_1\xi_2\xi_3$中设有另一函数$v_\xi\left(\xi_1,\xi_2,\xi_3\right)$,根据函数内积的定义并应用格林第一定理计算内积

$$\begin{aligned}\left(Tu_\xi,v_\xi\right)&=\iiint_{V_\xi}\underbrace{-\left(\nabla_\xi\cdot\nabla_\xi u_\xi\right)v_\xi}_{\text{应用格林第一定理}}\mathrm{d}V_\xi=\iiint_{V_\xi}\left(\nabla_\xi u_\xi\cdot\nabla_\xi v_\xi\right)\mathrm{d}V_\xi-\iiint_{V_\xi}\nabla_\xi\cdot\left(v_\xi\nabla_\xi u_\xi\right)\mathrm{d}V_\xi\\&=\iiint_{V_\xi}\left(\nabla_\xi u_\xi\cdot\nabla_\xi v_\xi\right)\mathrm{d}V_\xi-\oiint_{S_\xi}\left(v_\xi\nabla_\xi u_\xi\right)\cdot\mathrm{d}\boldsymbol{S}_\xi\\&=\iiint_{V_\xi}\left(\nabla_\xi u_\xi\cdot\nabla_\xi v_\xi\right)\mathrm{d}V_\xi-\oiint_{S_\xi}v_\xi\left(\nabla_\xi u_\xi\right)_{n_\xi}\mathrm{d}S_\xi=\iiint_{V_\xi}\left(\nabla_\xi u_\xi\cdot\nabla_\xi v_\xi\right)\mathrm{d}V_\xi-\oiint_{S_\xi}v_\xi\frac{\partial u_\xi}{\partial n_\xi}\mathrm{d}S_\xi,\end{aligned} \tag{4}$$

上式中,$\boldsymbol{n}_\xi$是矢量面元$\mathrm{d}\boldsymbol{S}_\xi$的单位外法线方向,$\dfrac{\partial u_\xi}{\partial n_\xi}$是函数$u_\xi\left(\xi_1,\xi_2,\xi_3\right)$沿着外法线的方向导数。

若令$u_\xi\big|_{\Gamma_\xi}=0$,或$\left.\dfrac{\partial u_\xi}{\partial n_\xi}\right|_{\Gamma_\xi}=0$($\Gamma_\xi$表示封闭曲面$S_\xi$的边界),即在第一类或第二类齐次边界条件下$\oiint_{S_\xi}v_\xi\dfrac{\partial u_\xi}{\partial n_\xi}\mathrm{d}S_\xi=0$,则计算内积式(4)为

$$\left(Tu_\xi,v_\xi\right)=\iiint_{V_\xi}\left(\nabla_\xi u_\xi\cdot\nabla_\xi\nu_\xi\right)\mathrm{d}V_\xi, \tag{5}$$

显然,将u_ξ和ν_ξ易位,上式(5)的计算结果不变,有$\left(Tu_\xi,v_\xi\right)=\left(u_\xi,Tv_\xi\right)$,即算子$T$是对称算子。

再令函数$\nu_\xi\left(\xi_1,\xi_2,\xi_3\right)=u_\xi\left(\xi_1,\xi_2,\xi_3\right)$,则计算内积式(5)为

$$\begin{aligned}\left(Tu_\xi,u_\xi\right)&=\iiint_{V_\xi}\left(\nabla_\xi u_\xi\cdot\nabla_\xi u_\xi\right)\mathrm{d}V_\xi=\iiint_{V_\xi}\left(\nabla_\xi u_\xi\right)^2\mathrm{d}V_\xi=\iiint_{V_\xi}\triangle_\xi u_\xi\mathrm{d}V_\xi\\&=\iiint_{V_\xi}\left[\left(\frac{\partial u_\xi}{\partial\xi_1}\right)^2+\left(\frac{\partial u_\xi}{\partial\xi_2}\right)^2+\left(\frac{\partial u_\xi}{\partial\xi_3}\right)^2\right]\mathrm{d}V_\xi\geqslant 0。\end{aligned} \tag{6}$$

因此,算子$T=-\triangle_\xi$是正定算子。

综合以上内积计算的结果式(5)和式(6),知算子T是对称正定算子。根据对称正定算子方程的变分原理知,泊松方程式(2)的电势解是使得构造的泛函

$$J\left[u_\xi\left(\xi_1,\xi_2,\xi_3\right)\right]=\left(Tu_\xi,u_\xi\right)-2\left(u_\xi,\rho_\xi\right) \tag{7}$$

取得极值时所满足的奥氏方程的解。

计算内积$\left(u_\xi,\rho_\xi\right)$并连带式(6)代入式(7)中,得到构造的泛函为

$$J_\xi\left[u_\xi\left(\xi_1,\xi_2,\xi_3\right)\right]=\iiint_{V_\xi}\left[\left(\frac{\partial u_\xi}{\partial\xi_1}\right)^2+\left(\frac{\partial u_\xi}{\partial\xi_2}\right)^2+\left(\frac{\partial u_\xi}{\partial\xi_3}\right)^2-2u_\xi\rho_\xi\right]\mathrm{d}V_\xi, \tag{8}$$

上式关于函数u_ξ的泛函是在电各向异性坐标系$O-\xi_1\xi_2\xi_3$中写出。由该泛函的核函数

$$F^\xi\left(\xi_1,\xi_2,\xi_3,u,u_{\xi_1},u_{\xi_2},u_{\xi_3}\right)=\left(\frac{\partial u_\xi}{\partial\xi_1}\right)^2+\left(\frac{\partial u_\xi}{\partial\xi_2}\right)^2+\left(\frac{\partial u_\xi}{\partial\xi_3}\right)^2-2u_\xi\rho_\xi \tag{9}$$

可知,泛函式(8)是关于单个函数u_ξ的具有3个独立变量ξ_1,ξ_2和ξ_3的一阶泛函。该一阶泛函取得极值时所满足的必要条件是函数u_ξ满足奥氏方程[10]

$$F^\xi_{u_\xi}-\frac{\partial}{\partial\xi_1}F^\xi_{\left(u_\xi\right)_{\xi_1}}-\frac{\partial}{\partial\xi_2}F^\xi_{\left(u_\xi\right)_{\xi_2}}-\frac{\partial}{\partial\xi_3}F^\xi_{\left(u_\xi\right)_{\xi_3}}=0。 \tag{10}$$

将关系式$F^\xi_{u_\xi}=\dfrac{\partial F^\xi}{\partial u_\xi}=-2\rho_\xi$,$\dfrac{\partial}{\partial\xi_1}F^\xi_{\left(u_\xi\right)_{\xi_1}}=\dfrac{\partial}{\partial\xi_1}\dfrac{\partial F^\xi}{\partial\left(\dfrac{\partial u_\xi}{\partial\xi_1}\right)}=\dfrac{\partial}{\partial\xi_1}\left(2\dfrac{\partial u_\xi}{\partial\xi_1}\right)=2\dfrac{\partial^2u_\xi}{\partial{\xi_1}^2}$,$\dfrac{\partial}{\partial\xi_2}F^\xi_{\left(u_\xi\right)_{\xi_2}}=2\dfrac{\partial^2u_\xi}{\partial{\xi_2}^2}$,$\dfrac{\partial}{\partial\xi_3}F^\xi_{\left(u_\xi\right)_{\xi_3}}=2\dfrac{\partial^2u_\xi}{\partial{\xi_3}^2}$代入式(10),得到奥氏方程

$$\frac{\partial^2u_\xi}{\partial{\xi_1}^2}+\frac{\partial^2u_\xi}{\partial{\xi_2}^2}+\frac{\partial^2u_\xi}{\partial{\xi_3}^2}=-\rho_\xi, \tag{11}$$

显然,泛函式(8)取得极值时满足的奥氏方程即是电各向异性坐标系下的泊松方程式(2)。

为了在主轴坐标系中研究各向异性电介质静电场泊松方程的变分问题,需将关系式$\dfrac{\partial u}{\partial\xi_1}=\dfrac{\partial u}{\partial x_1}\dfrac{\partial x_1}{\partial\xi_1}=\sqrt{\varepsilon_{11}}\dfrac{\partial u}{\partial x_1}$,$\dfrac{\partial u}{\partial\xi_2}=\dfrac{\partial u}{\partial x_2}\dfrac{\partial x_2}{\partial\xi_2}=\sqrt{\varepsilon_{22}}\dfrac{\partial u}{\partial x_2}$,$\dfrac{\partial u}{\partial\xi_3}=\dfrac{\partial u}{\partial x_3}\dfrac{\partial x_3}{\partial\xi_3}=\sqrt{\varepsilon_{33}}\dfrac{\partial u}{\partial x_3}$代入到电各向异性坐标系中泛函的核函数式(9)中得到各向异性电介质主轴坐标系$O-x_1x_2x_3$中的核函数

$$F\left(x_1,x_2,x,u,u_{x_1},u_{x_2},u_{x_3}\right)=\varepsilon_{11}\left(\frac{\partial u}{\partial x_1}\right)^2+\varepsilon_{22}\left(\frac{\partial u}{\partial x_2}\right)^2+\varepsilon_{33}\left(\frac{\partial u}{\partial x_3}\right)^2-2u\rho。 \tag{12}$$

由核函数式(12)可写出在主轴坐标系$O-x_1x_2x_3$中的泛函为

$$J\left[u\left(x_1,x_2,x_3\right)\right]=\iiint_{V_\xi}\left[\varepsilon_{11}\left(\frac{\partial u}{\partial x_1}\right)^2+\varepsilon_{22}\left(\frac{\partial u}{\partial x_2}\right)^2+\varepsilon_{33}\left(\frac{\partial u}{\partial x_3}\right)^2-2u\rho\right]\mathrm{d}V,\tag{13}$$

显然,泛函式(13)取得极值时所满足的奥氏方程的形式即是主轴坐标系$O-x_1x_2x_3$中泊松方程的具体形式式(1)。

综合以上变分原理分析,可知在电各向异性坐标系和主轴坐标系中,泊松方程第一和第二边值问题转化为与之等价的泊松方程的变分问题分别是

$$\begin{cases}J_\xi\left[u_\xi\left(\xi_1,\xi_2,\xi_3\right)\right]=\iiint_{V_\xi}\left[\left(\frac{\partial u_\xi}{\partial\xi_1}\right)^2+\left(\frac{\partial u_\xi}{\partial\xi_2}\right)^2+\left(\frac{\partial u_\xi}{\partial\xi_3}\right)^2-2u_\xi\rho_\xi\right]\mathrm{d}V_\xi\\ u_\xi\big|_{\Gamma_\xi}=0\text{ 或 }\frac{\partial u_\xi}{\partial n_\xi}\bigg|_{\Gamma_\xi}=0\end{cases}\tag{14}$$

和

$$\begin{cases}J\left[u\left(x_1,x_2,x_3\right)\right]=\iiint_{V}\left[\varepsilon_{11}\left(\frac{\partial u}{\partial x_1}\right)^2+\varepsilon_{22}\left(\frac{\partial u}{\partial x_2}\right)^2+\varepsilon_{33}\left(\frac{\partial u}{\partial x_3}\right)^2-2u\rho\right]\mathrm{d}V\\ u\big|_{\Gamma}=0\text{ 或 }\frac{\partial u}{\partial n}\bigg|_{\Gamma}=0,\end{cases}\tag{15}$$

式(15)中,$\frac{\partial u}{\partial n}$是函数$u\left(x_1,x_2,x_3\right)$沿着外法线$n$的方向导数,$\Gamma$表示封闭曲面$S$的边界。

由泊松方程的变分问题可知,在静电场有源的情况下泊松方程式(1)和式(2)分别是式(14)和(15)中的泛函取得极值时所满足的必要条件,这是从泛函的视角上进一步加深对电各向异性介质静电场泊松方程意义的理解。若静电场是无源的情况,则令$\rho_\xi=0$或$\rho=0$代入式(14)或式(15)中,即可将电各向异性介质静电场的拉普拉斯方程第一和第二边值问题转化与之等价的变分问题。

3 各向异性电介质静电场的能量密度泛函

泊松方程变分问题中的泛函式(15)所表达的物理含义表面上不明显,但只要稍作改造会发现该泛函实质上表达了各向异性电介质静电场的能量概念。设各向异性电介质主轴坐标系$O-x_1x_2x_3$中x_1、x_2和x_3方向的三个单位基矢量分别为$\boldsymbol{i}_1$、$\boldsymbol{i}_2$和$\boldsymbol{i}_3$,记主轴坐标系为$\boldsymbol{\chi}=(\boldsymbol{i}_1\quad\boldsymbol{i}_2\quad\boldsymbol{i}_3)$,$\boldsymbol{\chi}^{\mathrm{T}}$为$\boldsymbol{\chi}$的转置形态。由静电场电场强度与电势的梯度关系,得

$$\begin{aligned}\boldsymbol{E}&=-\nabla u\left(x_1,x_2,x_3\right)=-\left(\frac{\partial u}{\partial x_1}\boldsymbol{i}_1+\frac{\partial u}{\partial x_2}\boldsymbol{i}_2+\frac{\partial u}{\partial x_3}\boldsymbol{i}_3\right)\\&=-\left(\frac{\partial u}{\partial x_1}\quad\frac{\partial u}{\partial x_2}\quad\frac{\partial u}{\partial x_3}\right)\begin{pmatrix}\boldsymbol{i}_1\\\boldsymbol{i}_2\\\boldsymbol{i}_3\end{pmatrix}=-\left(\frac{\partial u}{\partial x_1}\quad\frac{\partial u}{\partial x_2}\quad\frac{\partial u}{\partial x_3}\right)\boldsymbol{\chi}^{\mathrm{T}},\end{aligned}\tag{16}$$

结合式(16),应用三维空间张量分析的矩阵方法[11]计算

$$\frac{1}{2}\boldsymbol{\varepsilon}\cdot\boldsymbol{E}\cdot\boldsymbol{E}=\frac{1}{2}\boldsymbol{\chi}\tilde{\varepsilon}\boldsymbol{\chi}^{\mathrm{T}}\cdot\boldsymbol{E}\cdot\boldsymbol{E}=\frac{1}{2}\boldsymbol{\chi}\begin{pmatrix}\varepsilon_{11}&0&0\\0&\varepsilon_{22}&0\\0&0&\varepsilon_{33}\end{pmatrix}\underbrace{\boldsymbol{\chi}^{\mathrm{T}}\cdot\boldsymbol{\chi}}_{=\bar{I}}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix}\cdot\boldsymbol{E}$$

$$=\frac{1}{2}\boldsymbol{\chi}\begin{pmatrix}\varepsilon_{11}&0&0\\0&\varepsilon_{22}&0\\0&0&\varepsilon_{33}\end{pmatrix}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix}\cdot\boldsymbol{E}=\frac{1}{2}\boldsymbol{\chi}\begin{pmatrix}\varepsilon_{11}E_1\\\varepsilon_{22}E_2\\\varepsilon_{33}E_3\end{pmatrix}\cdot\boldsymbol{E}=\frac{1}{2}\left(\varepsilon_{11}E_1\quad\varepsilon_{22}E_2\quad\varepsilon_{33}E_3\right)\underbrace{\boldsymbol{\chi}^{\mathrm{T}}\cdot\boldsymbol{\chi}}_{=\bar{I}}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}\varepsilon_{11}E_1 & \varepsilon_{22}E_2 & \varepsilon_{33}E_3\end{pmatrix}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix} = \frac{1}{2}\left(\varepsilon_{11}E_1^2 + \varepsilon_{22}E_2^2 + \varepsilon_{33}E_3^2\right)$$

$$= \frac{1}{2}\left[\varepsilon_{11}\left(\frac{\partial u}{\partial x_1}\right)^2 + \varepsilon_{22}\left(\frac{\partial u}{\partial x_2}\right)^2 + \varepsilon_{33}\left(\frac{\partial u}{\partial x_3}\right)^2\right], \tag{17}$$

上式中,$\boldsymbol{\varepsilon}$是各向异性电介质的介电常数张量,$\tilde{\varepsilon}$是$\boldsymbol{\varepsilon}$在主轴坐标系中的伴随矩阵,$\tilde{I}$是单位矩阵,E_1、E_2和E_3分别表示场强$\boldsymbol{E}$在主轴坐标系中沿$\boldsymbol{i}_1$、$\boldsymbol{i}_2$和$\boldsymbol{i}_3$轴的分量。

设泛函$J_E\left[u\left(x_1,x_2,x_3\right)\right] = \frac{1}{2}J\left[u\left(x_1,x_2,x_3\right)\right]$,再将式(17)代入式(15)的泛函中,得

$$J_E\left[u\left(x_1,x_2,x_3\right)\right] = \frac{1}{2}J\left[u\left(x_1,x_2,x_3\right)\right]$$

$$= \iiint_V \frac{1}{2}\left[\varepsilon_{11}\left(\frac{\partial u}{\partial x_1}\right)^2 + \varepsilon_{22}\left(\frac{\partial u}{\partial x_2}\right)^2 + \varepsilon_{33}\left(\frac{\partial u}{\partial x_3}\right)^2 - u\rho\right]\mathrm{d}V = \iiint_V\left(\frac{1}{2}\boldsymbol{\varepsilon}\cdot\boldsymbol{E}\cdot\boldsymbol{E} - u\rho\right)\mathrm{d}V$$

$$= \iiint_V\left[\frac{1}{2}(\boldsymbol{\varepsilon}\cdot\boldsymbol{E})\cdot\boldsymbol{E} - u\rho\right]\mathrm{d}V = \iiint_V\left(\frac{1}{2}\boldsymbol{D}\cdot\boldsymbol{E} - u\rho\right)\mathrm{d}V, \tag{18}$$

上式的计算过程中应用了各向异性电介质电位移矢量公式$\boldsymbol{D} = \boldsymbol{\varepsilon}\cdot\boldsymbol{E}$[1]。

定义各向异性电介质有源静电场的能量密度$\mathcal{W}_E' = \frac{1}{2}\boldsymbol{D}\cdot\boldsymbol{E} - u\rho$,则式(18)写成

$$J_E\left[u\left(x_1,x_2,x_3\right)\right] = \iiint_V\left(\frac{1}{2}\mathcal{W}_E' - u\rho\right)\mathrm{d}V, \tag{19}$$

是描述各向异性电介质有源静电场的能量密度泛函(或称为能量积分)。显然,因为泛函$J_E[u] = \frac{1}{2}J[u]$具有线性关系,可知使能量密度泛函式(19)取得极值时所满足的必要条件是电势$u\left(x_1,x_2,x_3\right)$满足泊松方程式(2)。若令$\rho = 0$,此时$\mathcal{W}_E' = \frac{1}{2}\boldsymbol{D}\cdot\boldsymbol{E}$表示的是各向异性电介质无源静电场的能量密度,$J_E\left[u\left(x_1,x_2,x_3\right)\right] = \frac{1}{2}\iiint_{V_\xi}\boldsymbol{D}\cdot\boldsymbol{E}\mathrm{d}V$是描述各向异性电介质无源静电场的能量泛函,要使该能量泛函取得极值时的必要条件是电势$u\left(x_1,x_2,x_3\right)$满足各向异性电介质静电场的拉普拉斯方程。

4 结论

先在电各向异性坐标系中研究各向异性电介质静电场的属性(如物理量或微分方程),再将该属性通过坐标变换到主轴坐标系中来描述,是常用的数学处理方法,该方法简便且实用。本文先在电各向异性坐标系$O-\xi_1\xi_2\xi_3$中讨论对称正定算子方程的变分原理,得到泛函的核函数式(9),再通过坐标变换将该核函数转化为在各向异性电介质主轴坐标系$O-x_1x_2x_3$中的核函数式(12),进一步写出泛函的表达式(13),最后得到由式(15)描述的在主轴坐标系中与泊松方程第一和第二边值问题等价的泊松方程的变分问题。

对泊松方程变分问题中泛函式(13)做进一步研究发现该泛函式具有静电场的能量意义。由泛函式(18)的核函数出发,定义了各向异性电介质静电场的能量密度概念,从而推导出能量密度泛函式(19)。在泛函的角度上得到了关于各向异性电介质静电场泊松方程的新理解,泊松方程式(1)是使能量密度泛函式(19)取得极值的必要条件,泊松方程的电势解是能量密度泛函取得极值的极值函数。

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