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无界域上带非线性阻尼和强阻尼随机波动方程解的性态

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摘　要:本文研究无界域上带非线性阻尼、强阻尼以及可加噪声的非自治随机波动方程随机吸引子的存在性。首先证明该方程组的解可以定义一个随机动力系统,然后对方程的解进行一致估计得到此随机动力系统D-拉回随机吸收集的存在性,最后利用空间分割的方法克服无界域上Sobolev嵌入缺乏紧性的困难并证得此随机动力系统的D-拉回渐近紧性,进而得到该动力系统随机吸引子的存在性。

关键词:强阻尼;波动方程;随机吸引子;渐近紧性;无界域
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The Behavior of Solutions to Stochastic Wave Equations with Nonlinear Damping and Strong Damping on Unbounded Domains

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Abstract: This article investigated the existence of random attractor for non-autonomous stochastic wave equation with nonlinear damping term and strong damping term, as well as additive noise on the unbounded domain. Firstly, it was proved that the solution of the equation can generate a random dynamical system, and then the uniform estimation of the solution was given to get the existence of D-pullback random absorbing set of the random dynamical system. Finally, the method of space division was used to overcome the non-compactness of Sobolev embedding on the unbounded domain and get the D-pullback asymptotical compactness of the random dynamical system, and then the existence of the random attractor of the dynamic system was obtained.

Keywords: strong damping; wave equations; random attractor; asymptotic compactness; unbounded domain

本文研究定义在$\mathbf{R}^n$上带有非线性阻尼、强阻尼以及可加噪声的非自治随机波动方程解的渐近行为:

$$u_{tt}+h\left(u_t\right)-\Delta u-\alpha\Delta u_t+\xi u+f\left(x,u\right)=g\left(x,t\right)+Q\left(x\right)\frac{\mathrm{d}W}{\mathrm{d}t},t>\tau,\tag{1}$$

初值为

$$u\left(\tau,x\right)=u_0\left(x\right),u_t\left(\tau,x\right)=u_1\left(x\right),x\in\mathbf{R}^n,\tau\in R,\tag{2}$$

其中,$1\leqslant n\leqslant 3$,ξ是正常数,$g\in L^2_{loc}\left(R,L^2\left(\mathbf{R}^n\right)\right)$,$Q\left(x\right)\in H^1\left(\mathbf{R}^n\right)$,$W$是定义在概率空间$\left(\Omega,F,P\right)$上的双边实值Wiener过程,其中$\Omega=\left\{\omega\in C\left(R,R\right):\omega\left(0\right)=0\right\}$,$F$为$\Omega$上的一个Borel σ-代数,P是相应的Wiener测度,假设非线性函数h、f满足以下条件:

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（H1）对任意的$x \in \mathbf{R}^n, u \in R$，有

$$\left|f(x,u)\right| \leqslant c_1|u|^{\gamma} + \varphi_1(x), \varphi_1(x) \in L^2(\mathbf{R}^n), \tag{3}$$

$$f(x,u)u - c_2F(x,u) \geqslant \varphi_2(x), \varphi_2(x) \in L^1(\mathbf{R}^n), \tag{4}$$

$$F(x,u) \geqslant c_3|u|^{\gamma+1} - \varphi_3(x), \varphi_3(x) \in L^1(\mathbf{R}^n), \tag{5}$$

$$\left|f_u(x,u)\right| \leqslant c_4|u|^{\gamma-1} + \varphi_4(x), \varphi_4(x) \in H^1(\mathbf{R}^n), \tag{6}$$

其中，$c_i(i = 1,2,3,4)$是正的常数；$F(x,u) = \int_0^u f(x,s)\mathrm{d}s$；当$n = 1,2$时，$\gamma \geqslant 1$，当$n = 3$时，$\gamma \in [1,3)$。

（H2）存在正常数β_1和β_2，使得

$$h(0) = 0, 0 < \beta_1 \leqslant h'(v) \leqslant \beta_2 < \infty。\tag{7}$$

（H3）假设非自治确定外力项满足条件：存在正常数σ，使得

$$\int_{-\infty}^{\tau} e^{\sigma s}\left\|g(\cdot,s)\right\|^2 \mathrm{d}s < \infty, \forall \tau \in R。\tag{8}$$

波动方程是一类重要的偏微分方程，具有广泛的物理背景，主要用于描述自然界中各种波动现象，例如声波、光波和无线电波等，为声学、电磁学和流体力学等领域的研究提供了强有力的科学理论依据。目前已有许多学者对波动方程进行了研究，如：文献[1-2]研究了有界区域上自治随机波动方程所确定的随机动力系统随机吸引子的存在性，文献[3-4]研究了带强阻尼波方程吸引子的存在性。文献[5-6]对非自治随机动力系统进行了研究，通过引入两个驱动动力系统，证明了非自治随机动力系统随机吸引子存在的充分必要条件。关于随机波动方程的其他研究参见文献[7-8]。目前还没有文献研究无界域上带有非线性阻尼以及强阻尼的随机波动方程。

本文将研究无界域上带有非线性阻尼以及强阻尼的随机波动方程。在无界域的情况下，由于Sobolev嵌入紧性的缺失，可以利用有界球上的一致估计以及无界域上的尾部一致小估计证明动力系统的渐近紧性，如文献[9]。本文采用对解的一致估计和区域分割的方法来克服$\mathbf{R}^n$上Sobolev嵌入缺乏紧性的困难，即在有界域上对相应的解进行渐近估计，对相应无界域上的解进行一致小估计，并结合解的分解估计得到了该随机动力系统的渐近紧性，进而得到随机吸引子的存在性。

本文中，$\|\cdot\|$和$(\cdot,\cdot)$分别表示$L^2(\mathbf{R}^n)$上的范数和内积，$\|\cdot\|_X$表示一般的Banach空间上的范数，字母c表示某一正常数，且其值可能每行各不相同，字母$c_i(i = 1,2,\cdots)$表示特定常数。

1 准备知识

令X是可分的Banach空间，(Ω,F,P)是概率空间，其中$\Omega = \{\omega \in C(R,R): \omega(0) = 0\}$，$F$是$\Omega$上紧开拓扑诱导的Borel σ-代数，P表示(Ω,F)上相应的Wiener测度。定义变换

$$\theta_t\omega(s) = \omega(t+s) - \omega(t), \omega \in \Omega, \forall t,s \in R,$$

则$\left(\Omega,F,P,(\theta_t)_{t \in R}\right)$为完备度量动力系统。下面引入一些关于随机动力系统以及随机吸引子的概念与基本结果，详见文献[5]。

定义1.1 称定义在X和R以及$\left(\Omega,F,P,(\theta_t)_{t \in R}\right)$上的映射$\Phi: R^+ \times R \times \Omega \times X \to X$为连续动力系统，若对任意的$\tau \in R, \omega \in \Omega, t,s \in R^+$，有

1）$\Phi(\cdot,\tau,\cdot,\cdot): R^+ \times \Omega \times X \to X$是$\left(B(R^+) \times F \times B(X), B(X)\right)$-可测的；

2）$\Phi(0,\tau,\omega_2,\cdot)$是$X$上的恒同映射；

3）$\Phi(t+s,\tau,\omega,\cdot) = \Phi(t,\tau+s,\theta_s\omega,\cdot) \circ \Phi(s,\tau,\omega,\cdot)$；

4）$\Phi(t,\tau,\omega,\cdot): X \to X$是连续的。

设$\tilde{D} = \{\tilde{D}(\tau,\omega): \tau \in R, \omega \in \Omega\}$是$X$的非空子集族，$D$表示集族满足一定条件的$\tilde{D}$的集合。

定义 1.2 称X上的非空有界子集族$\tilde{D}=\left(\tilde{D}(\tau,\omega):\tau\in R,\omega\in\Omega\right)$是缓增的随机集族，如果对任意的$c>0,\tau\in R,\omega\in\Omega$，有

$$\lim_{t\to\infty}e^{ct}\left\|\tilde{D}(\tau+t,\theta_t\omega)\right\|=0,\tag{9}$$

其中$\left\|\tilde{D}\right\|=\sup_{x\in D}\|x\|$。

给定$\tilde{D}\in D$，称$\Omega\left(\tilde{D}\right)=\left(\Omega\left(\tilde{D},\tau,\omega\right):\tau\in R,\omega\in\Omega\right)$是$\tilde{D}$的$\Omega-$极限集，其中

$$\Omega\left(\tilde{D},\tau,\omega\right)=\bigcap_{r\geqslant 0}\overline{\bigcup_{t\geqslant r}\Phi\left(t,\tau-t,\theta_{-t}\omega,\tilde{D}\left(\tau-t,\theta_{-t}\omega\right)\right)}。\tag{10}$$

定义 1.3 称动力系统Φ在X上是$D-$拉回渐近紧的，如果对任意的$\tau\in R,\omega\in\Omega$，当$t_n\to\infty$时，

$$x_n\in\tilde{D}\left(\tau-t_n,\theta_{-t_n}\omega\right),\left\{\tilde{D}\left(\tau,\omega\right):\tau\in R,\omega\in\Omega\right\}\in D,$$

序列$\left\{\Phi\left(t_n,\tau-t_n,\theta_{-t_n}\omega,x_n\right)\right\}_{n=1}^{\infty}$在$X$中有收敛子列。

定义 1.4 称$K=\left\{K\left(\tau,\omega\right):\tau\in R,\omega\in\Omega\right\}\in D$是$\Phi$的$D-$拉回随机吸收集，若对任意的$t\in R^+,\tau\in R$，$\omega\in\Omega,\tilde{D}\in D$，存在$T=T\left(\tilde{D},\tau,\omega\right)>0$，使得对任意的$t\geqslant T$，有

$$\Phi\left(t,\tau-t,\theta_{-t}\omega,\tilde{D}\left(\tau-t,\theta_{-t}\omega\right)\right)\subseteq K\left(\tau,\omega\right)。\tag{11}$$

定义 1.5 称$A=\left\{A\left(\tau,\omega\right):\tau\in R,\omega\in\Omega\right\}\in D$是$\Phi$的$D-$拉回随机吸引子，若对任意的$t\in R^+,\tau\in R$，$\omega\in\Omega$，有

1）$A\left(\tau,\omega\right)$在$X$上是紧的且$A$关于$F$在$\Omega$中是可测的；

2）A是不变的，即$\Phi\left(t,\tau,\omega,A\left(\tau,\omega\right)\right)=A\left(t+\tau,\theta_t\omega\right)$；

3）A吸引D中的任意集合，即对任意的$\tilde{D}=\left\{\tilde{D}\left(\tau,\omega\right):\tau\in R,\omega\in\Omega\right\}\in D$，

$$\lim_{t\to\infty}d_H\left(\Phi\left(t,\tau-t,\theta_{-t}\omega,\tilde{D}\left(\tau-t,\theta_{-t}\omega\right)\right),A\left(\tau,\omega\right)\right)=0,\tag{12}$$

其中d_H是X上的Hausdorff半距离。

定理 1[5] 令D是X上包含闭的非空子集族，若Φ在X上是$D-$拉回渐近紧的，且在D上存在一个闭可测$D-$拉回随机吸引集K，那么，对任意的$\tau\in R,\omega\in\Omega$，$\Phi$在$D$上存在唯一的$D-$拉回随机吸引子$A$有

$$A\left(\tau,\omega\right)=\Omega\left(K,\tau,\omega\right)=\bigcup_{\tilde{D}\in D}\Omega\left(\tilde{D},\tau,\omega\right)。$$

2 随机动力系统

本节证明式（1）~（2）的解可以生成随机动力系统，考虑一维Ornstein - Uhlenbeck方程

$$\mathrm{d}z+z\mathrm{d}t=\mathrm{d}W\left(t\right),\tag{13}$$

令$\tilde{A}=-\Delta$，由文献[5]可知，$z\left(\theta_t\omega\right):=-\int_{-\infty}^{0}e^s\left(\theta_t\omega\right)\left(s\right)\mathrm{d}s,t\in R,\omega\in\Omega$是式（13）的稳态解，且映射$t\to z\left(\theta_t\omega\right)$关于$t$连续，并且有

$$\lim_{t\to+\infty}e^{-\gamma t}\left|z\left(\theta_{-t}\omega\right)\right|=0,\forall\gamma>0。$$

令$v=u_t+\varepsilon u-Q\left(x\right)z\left(\theta_t\omega\right),t\geqslant\tau,\tau\in R$，则式（1）~（2）等价于下列系统，

$$\frac{\mathrm{d}u}{\mathrm{d}t}=v-\varepsilon u+Q\left(x\right)z\left(\theta_t\omega\right),\tag{14}$$

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}t}=&\alpha\Delta v+\left(1-\alpha\varepsilon\right)\Delta u+\varepsilon v-\left(\xi+\varepsilon^2\right)u-h\left(v-\varepsilon u+Q\left(x\right)z\left(\theta_t\omega\right)\right)+\alpha z\left(\theta_t\omega\right)\Delta Q\left(x\right)\\&+\left(\varepsilon+1\right)z\left(\theta_t\omega\right)Q\left(x\right)-f\left(x,u\right)+g\left(x,t\right),\end{aligned}\tag{15}$$

初值为

$$u(\tau,\tau,x)=u_0(x),v(\tau,\tau,x)=v_0(x),\tag{16}$$

即

$$\begin{cases}\phi+\Lambda\phi=G\left(\phi,\theta_t\omega,t\right),t\geqslant\tau\\ \phi(\tau,\omega)=\phi_0(\omega)=\left(u_0,u_{1,0}+\varepsilon u_0(x)-z\left(\theta_t\omega\right)Q(x)\right)\end{cases},\tag{17}$$

其中,

$$\phi=(u,v)^{\mathrm{T}},\Lambda=\begin{pmatrix}\varepsilon I & -I\\ \tilde{A}(1-\alpha\varepsilon)+\left(\xi+\varepsilon^2\right)I & \alpha\tilde{A}-\varepsilon I\end{pmatrix},$$

$$G\left(\phi,\theta_t\omega,t\right)=\begin{pmatrix}Q(x)z\left(\theta_t\omega\right)\\ -f(x,u)+g(x,t)+(1+\varepsilon)z\left(\theta_t\omega\right)Q(x)+\alpha z\left(\theta_t\omega\right)\Delta Q(x)-h\left(v-\varepsilon u+zQ(x)\right)\end{pmatrix}。$$

令$E\left(\mathbf{R}^n\right)=H^1\left(\mathbf{R}^n\right)\times L^2\left(\mathbf{R}^n\right)$,其范数为

$$\|\Phi\|_{H^1\left(\mathbf{R}^n\right)\times L^2\left(\mathbf{R}^n\right)}=\left(\|\nabla u\|^2+\|u\|^2+\|v\|^2\right)^{1/2},\Phi=(u,v)^{\mathrm{T}}\in E\left(\mathbf{R}^n\right)。\tag{18}$$

引理2.1[4] 对任意$\varepsilon>0$,有缓增随机变量$k:\Omega\mapsto R^+$,使得对任意的$t\in R$, $\omega\in\Omega$,

$$\left\|Q(x)z\left(\theta_t\omega\right)\right\|\leqslant e^{\varepsilon|t|}k(\omega)\|Q\|,$$

$$\left\|\nabla Q(x)z\left(\theta_t\omega\right)\right\|\leqslant e^{\varepsilon|t|}k(\omega)\|\nabla Q\|,$$

$$\left\|\Delta Q(x)z\left(\theta_t\omega\right)\right\|\leqslant e^{\varepsilon|t|}k(\omega)\|\Delta Q\|,$$

其中

$$e^{-\varepsilon|t|}k(\omega)\leqslant k\left(\theta_t\omega\right)\leqslant e^{\varepsilon t}k(\omega)。$$

如同文献[5]中的证明,有如下引理:

引理2.2 令$\phi\left(t+\tau,\tau,\theta_{-\tau}\omega,\phi_0\right)=\left(u\left(t+\tau,\tau,\theta_{-\tau}\omega,u_0\right),v\left(t+\tau,\tau,\theta_{-\tau}\omega,v_0\right)\right)^{\mathrm{T}}$,其中$\phi_0=\left(u_0,v_0\right)^{\mathrm{T}}$,且式(3)~(7)成立,则对任意的$\omega\in\Omega,\tau\in R,\phi\left(\tau,\tau,\omega,\phi_0\right)=\phi_0\in E\left(\mathbf{R}^n\right)$,式(14)~(16)在$E\left(\mathbf{R}^n\right)$上存在唯一解$\phi\left(\cdot,\tau,\omega,\phi_0\right)\in C\left([\tau,+\infty),E\left(\mathbf{R}^n\right)\right)$,且该解连续依赖于初值$\phi_0$,此时式(1)~(2)的解生成连续的随机动力系统$\Phi:R^+\times R\times\Omega\times E\left(\mathbf{R}^n\right)\rightarrow E\left(\mathbf{R}^n\right)$,

$$\Phi\left(t,\tau,\omega,\phi_0\right)=\phi\left(t+\tau,\tau,\theta_{-\tau}\omega,\phi_0\right),\forall\left(t,\tau,\omega,\phi_0\right)\in R^+\times R\times\Omega\times E\left(\mathbf{R}^n\right)。\tag{19}$$

3 解的一致估计

为证明D-拉回随机吸收集的存在性以及随机动力系统Φ的D-拉回渐近紧性,下面对式(14)~(16)的解进行一致估计。

引理3.1 设$Q(x)\in H^1\left(\mathbf{R}^n\right)$,假设(H1)~(H3)成立,则对任意的$\tau\in R,\omega\in\Omega$,存在一个缓增随机向量$M_0(\omega)$,使得对任意的$B\in D(E)$,存在$T=T(\tau,\omega,B)\geqslant0$,使得当$t\geqslant T$时,有

$$\left\|\Phi\left(t,\tau-t,\theta_{-\tau}\omega,\phi_{\tau-t}\left(\theta_{-\tau}\omega\right)\right)\right\|_E\leqslant M_0(\omega),\forall t\geqslant T(\tau,\omega,B)。\tag{20}$$

证明 将式(17)与$\phi(r)$在$E\left(R^n\right)$中作内积可得

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi(r)\|_E^2+(\Lambda\phi,\phi)_E\\&=\left(z\left(\theta_{r-\tau}\omega\right)Q(x),u\right)_1+\left(f(x,u),v\right)+\left(g(x,r)+(1+\varepsilon)Q(x)z\left(\theta_{r-\tau}\omega\right),v\right)\\&+\left(\alpha z\left(\theta_{r-\tau}\omega\right)\Delta Q(x),v\right)-\left(h\left(v-\varepsilon u+Q(x)z\left(\theta_{r-\tau}\omega\right)\right),v\right)。\end{aligned}\tag{21}$$

现对式(21)逐项进行估计。应用Cauchy不等式可得

$$\left(z\left(\theta_{r-\tau}\omega\right)Q(x),u\right)_1 \leqslant \frac{2}{\varepsilon}\left\|z\left(\theta_{r-\tau}\omega\right)Q(x)\right\|_1^2+\frac{\varepsilon}{8}\|u\|_1^2, \tag{22}$$

$$\left(\alpha z\left(\theta_{r-\tau}\omega\right)\Delta Q(x),v\right) \leqslant \alpha\left\|z\left(\theta_{r-\tau}\omega\right)\nabla Q(x)\right\|_1^2+\frac{\alpha}{4}\|\nabla v\|_1^2, \tag{23}$$

$$\left(g(x,t)+(1+\varepsilon)z\left(\theta_{r-\tau}\omega\right)Q(x),v\right) \leqslant \frac{1}{\alpha}\left[\left\|g(x,r)\right\|^2+(1+\varepsilon)^2\left\|z\left(\theta_{r-\tau}\omega\right)Q(x)\right\|^2\right]+\frac{\alpha}{4}\|v\|^2。 \tag{24}$$

由式(14)可知

$$\left(f(x,u),v\right)=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^n}F(x,u)\mathrm{d}x+\varepsilon\left(f(x,u),u\right)-\left(f(x,u),z\left(\theta_{r-\tau}\omega\right)Q(x)\right)。 \tag{25}$$

由文献[4]可知

$$\left(\Lambda\phi,\phi\right)_E \geqslant \frac{\varepsilon}{2}\left(\|u\|_1^2+\|v\|^2\right)+\frac{\alpha}{2}\|v\|^2。 \tag{26}$$

应用假设(H1)以及Cauchy不等式可得

$$\left(f(x,u),v\right) \geqslant c_2\int_{\mathbf{R}^n}F(x,u)\mathrm{d}x+\int_{\mathbf{R}^n}\varphi_2\mathrm{d}x,$$

$$\begin{aligned}\left(f(x,u),z\left(\theta_{r-\tau}\omega\right)Q(x)\right) &\leqslant \int_{\mathbf{R}^n}\left(c_1|u|^\gamma+\varphi_1\right)\cdot\left|z\left(\theta_{r-\tau}\omega\right)Q(x)\right|\mathrm{d}x\\ &\leqslant \frac{1}{2}\left\|\varphi_1\right\|^2+\frac{1}{2}\left\|z\left(\theta_{r-\tau}\omega\right)Q(x)\right\|^2+\frac{\varepsilon c_1}{2c_3}\int_{\mathbf{R}^n}F(x,u)\mathrm{d}x\\ &\quad+\frac{\varepsilon c_1}{2c_3}\int_{\mathbf{R}^n}\varphi_3(x)\mathrm{d}x+c\left\|z\left(\theta_{r-\tau}\omega\right)Q(x)\right\|_{H^1}^{\gamma+1}。\end{aligned}$$

由假设(H2)可得

$$\begin{aligned}\left(h\left(v-\varepsilon u+Q(x)z\right),v\right) &= \left(h\left(v-\varepsilon u+Qz\right)-h(0),v\right)\\ &= h'(\vartheta)\left(v-\varepsilon u+Q(x)z,v\right)\\ &\geqslant \beta_1\|v\|^2+\beta_1\left(Q(x)z\left(\theta_t\omega\right),v\right)-\beta_2\varepsilon(u,v),\end{aligned} \tag{27}$$

其中 ϑ 在0和 $v-\varepsilon u+Qz$ 之间。$(\varepsilon+1)\left(Q(x)z\left(\theta_t\omega\right),v\right)-\beta_1\left(Q(x)z\left(\theta_t\omega\right),v\right)=\left(\varepsilon-\beta_1+1\right)\left(Qz\left(\theta_t\omega\right),v\right) \leqslant \frac{\alpha}{4}\|v\|_1^2+\frac{1}{\alpha}\left\|Qz\left(\theta_t\omega\right)\right\|^2$。将式(22)~(27)代入式(21)，对 $r \geqslant \tau-t$，有

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|\phi(r)\right\|_E^2+2\int_{\mathbf{R}^n}F(x,u)\mathrm{d}x\right]+\frac{\varepsilon}{4}\left\|\phi(r)\right\|_E^2+\frac{\alpha}{4}\|v\|_1^2+\frac{\varepsilon c_2}{2}\int_{\mathbf{R}^n}F(x,u)\mathrm{d}x\\ &\leqslant c_3\left(1+\left\|z\left(\theta_{r-\tau}\right)Q(x)\right\|^2+\left\|z\left(\theta_{r-\tau}\right)\nabla Q(x)\right\|^2+\left\|z\left(\theta_{r-\tau}\right)Q(x)\right\|_{H^1}^{\gamma+1}\right)+\frac{1}{\alpha}\|g\|^2。\end{aligned} \tag{28}$$

在式(28)两边同乘 $e^{\sigma t}$，并在 $[\tau-t,\tau]$ 上积分，可得

$$\begin{aligned}&e^{\sigma\tau}\left(\left\|\phi\left(\tau,\tau-t,\omega,\phi_0\right)\right\|_{E\left(\mathbf{R}^n\right)}^2+2\int_{\mathbf{R}^n}F\left(x,u\left(\tau,\tau-t,\omega,u_0\right)\right)\mathrm{d}x\right)+\frac{\varepsilon}{2}\int_{\tau-t}^{\tau}e^{\sigma s}\left\|\phi\left(s,\tau-t,\omega,\phi_0\right)\right\|_{E\left(\mathbf{R}^n\right)}^2\mathrm{d}s\\ &\quad+\frac{\alpha}{2}\int_{\tau-t}^{\tau}\left\|v\left(s,\tau-t,\omega,v_0\right)\right\|_1^2\mathrm{d}s\\ &\leqslant e^{\sigma(\tau-t)}\left(\left\|\phi_0\right\|_{E\left(\mathbf{R}^n\right)}^2+2\int_{\mathbf{R}^n}F\left(x,u_0\right)\mathrm{d}x\right)+c\int_{\tau-t}^{\tau}e^{\sigma s}\left(1+\left\|z\left(\theta_s\omega\right)Q\right\|^2+\left\|z\left(\theta_s\omega\right)\nabla Q\right\|^2+\left\|z\left(\theta_s\omega\right)Q\right\|_{H^1}^{\gamma+1}\right)\mathrm{d}s\\ &\quad+\frac{2}{\alpha}\int_{\tau-t}^{\tau}e^{\sigma s}\left\|g(x,s)\right\|^2\mathrm{d}s。\end{aligned} \tag{29}$$

用 $\theta_{-\tau}\omega$ 代替 ω，则由式(29)可得

$$\begin{aligned}&\left\|\phi\left(\tau,\tau-t,\theta_{-\tau}\omega,\phi_0\right)\right\|_{E\left(\mathbf{R}^n\right)}^2+2\int_{\mathbf{R}^n}F\left(x,u\left(\tau,\tau-t,\theta_{-\tau}\omega,u_0\right)\right)\mathrm{d}x\\ &\quad+\frac{\varepsilon}{2}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\left\|\phi\left(s,\tau-t,\theta_{-\tau}\omega,\phi_0\right)\right\|_{E\left(\mathbf{R}^n\right)}^2\mathrm{d}s+\frac{\alpha}{2}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\left\|v\left(s,\tau-t,\theta_{-\tau}\omega,v_0\right)\right\|_1^2\mathrm{d}s\end{aligned}$$

$$\leqslant c\int_{-t}^{0} e^{\sigma s}\left(1+\left\|z\left(\theta_s\omega\right)Q\right\|^2+\left\|z\left(\theta_s\omega\right)\nabla Q\right\|^2+\left\|z\left(\theta_s\omega\right)Q\right\|_{H^1}^{\gamma+1}\right)\mathrm{d}s$$

$$+\frac{2}{\alpha}\int_{-\infty}^{\tau} e^{\sigma(s-\tau)}\left\|g(x,s)\right\|^2\mathrm{d}s+e^{-\sigma t}\left(\left\|\phi_0\right\|_{E(\mathbf{R}^n)}^2+2\int_{\mathbf{R}^n}F\left(x,u_0\right)\mathrm{d}x\right)。\tag{30}$$

令$\varepsilon=\dfrac{\sigma}{2(\gamma+1)}$,由引理2.1可得

$$\int_{-t}^{0} e^{\sigma s}\left(1+\left\|z\left(\theta_s\omega\right)Q\right\|^2+\left\|z\left(\theta_s\omega\right)\nabla Q\right\|^2+\left\|z\left(\theta_s\omega\right)Q\right\|_{H^1}^{\gamma+1}\right)\mathrm{d}s$$

$$\leqslant\int_{-\infty}^{0} e^{\sigma s/2}\left(k^2(\omega)\left(\left\|\nabla Q(x)\right\|^2+\left\|Q(x)\right\|^2\right)+k^{\gamma+1}(\omega)\left(\left\|\nabla Q(x)\right\|^{\gamma+1}+\left\|Q(x)\right\|^{\gamma+1}\right)\right)\mathrm{d}s<\infty。\tag{31}$$

应用式(3)~(4),有

$$\int_{\mathbf{R}^n}F\left(x,u_0\right)\mathrm{d}x\leqslant c\left(1+\left\|u_0\right\|^2+\left\|u_0\right\|_{H^1}^{\gamma+1}\right),\tag{32}$$

因为$\phi_0=\left(u_0,v_0\right)^{\mathrm{T}}\in\tilde{D}\left(\tau-t,\theta_{-t}\omega\right)$,故有

$$\lim_{t\to+\infty}e^{-\sigma t}\left(\left\|\phi_0\right\|_{E(\mathbf{R}^n)}^2+2\int_{\mathbf{R}^n}F\left(x,u_0\right)\mathrm{d}x\right)=0,\tag{33}$$

所以存在$T=T(\tau,\omega,B)>0$,使得当$t\geqslant T$时,有

$$e^{-\sigma t}\left(\left\|\phi_0\right\|_{E(\mathbf{R}^n)}^2+2\int_{\mathbf{R}^n}F\left(x,u_0\right)\mathrm{d}x\right)\leqslant\varepsilon。\tag{34}$$

令 $M_0=c+c\int_{-t}^{0} e^{\sigma s}\left(1+\left\|z\left(\theta_s\omega\right)Q\right\|^2+\left\|z\left(\theta_s\omega\right)\nabla Q\right\|^2+\left\|z\left(\theta_s\omega\right)Q\right\|_{H^1}^{\gamma+1}\right)\mathrm{d}s+c\int_{-x}^{t} e^{\sigma(s-\tau)}\left\|g(x,s)\right\|^2\mathrm{d}s$, 因此,由假设(H3)、式(32)和式(34)可知结论成立。

由上述结论可得以下引理。

引理3.2 设$Q(x)\in H^1\left(\mathbf{R}^n\right)$,假设(H1)~(H3)成立,则定义在$E\left(\mathbf{R}^n\right)$上的连续动力系统$\Phi\left(t,\tau,\omega,\phi_0\right)$有随机拉回吸收集$B=\left\{B(\tau,\omega):\tau\in R,\omega\in\Omega\right\}\in D$,其中$B(\tau,\omega)$定义为

$$B(\tau,\omega)=\left\{\phi\in E\left(\mathbf{R}^n\right):\|\phi\|_{E(\mathbf{R}^n)}\leqslant M_0(\omega)\right\}。$$

令ρ是光滑函数,对任意的$s\in R$,有$0\leqslant\rho(s)\leqslant1$,且

$$\rho(s)=\begin{cases}0,0\leqslant s\leqslant1\\ 1,|s|\geqslant2\end{cases},\tag{35}$$

则存在正常数μ_1,μ_2,使得对任意的$s\in R$,$\left|\rho'(s)\right|\leqslant\mu_1$,$\left|\rho''(s)\right|\leqslant\mu_2$。给定$r\geqslant1$,记$Q_r=\left\{x\in\mathbf{R}^n:|x|\leqslant r\right\}$及$Q_r$的补集$\mathbf{R}^n\backslash Q_r$。为得到$\mathbf{R}^n$上解的渐近紧性,先证如下引理。

引理3.3 假设(H1)~(H3)成立,则对任意的$\varepsilon>0,\tau\in R,\omega\in\Omega$,存在$\hat{T}>0,\hat{R}\geqslant1$,使得当$t\geqslant\hat{T},r\geqslant\hat{R}$时,有

$$\left\|\Phi\left(t,\theta_{-t}\omega,\phi_0\left(\theta_{-t}\omega\right)\right)\right\|_{E(\mathbf{R}^n\backslash Q_r)}^2\leqslant\varepsilon。\tag{36}$$

证明 将式(15)与$\rho\left(\dfrac{|x|^2}{r^2}\right)v$在$E\left(\mathbf{R}^n\right)$上作内积,可得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)|v|^2\mathrm{d}x=\alpha\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)v\cdot\Delta v\mathrm{d}x+(1-\alpha\varepsilon)\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)v\cdot\Delta u\mathrm{d}x$$

$$-\left(\xi+\varepsilon^2\right)\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)uv\mathrm{d}x-\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)h\left(v-\varepsilon u+Q(x)z\left(\theta_t\omega\right)\right)v\mathrm{d}x$$

$$-\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)f(x,u)v\mathrm{d}x+\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)g(x,t)v\mathrm{d}x$$

$$+(1+\varepsilon)\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)v\cdot Qz(\theta_t\omega)\mathrm{d}x+\alpha\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)z(\theta_t\omega)\cdot\Delta Q\cdot v\mathrm{d}x,\tag{37}$$

注意到

$$\alpha\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)v\cdot\Delta v\mathrm{d}x=-\alpha\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)|\nabla v|^2\mathrm{d}x,\tag{38}$$

$$\alpha\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)v\cdot\Delta v\mathrm{d}x=-\alpha\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)|\nabla v|^2\mathrm{d}x,$$

$$\leqslant\frac{\sqrt{2}\,c(\alpha\varepsilon-1)}{r}\left(\|\nabla u\|^2+\|v\|^2\right)+\frac{\alpha\varepsilon-1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)|\nabla u|^2\mathrm{d}x$$

$$+\varepsilon(\alpha\varepsilon-1)\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)|\nabla u|^2\mathrm{d}x,\tag{39}$$

$$\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)uv\mathrm{d}x\geqslant\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)|u|^2\mathrm{d}x+\frac{\varepsilon}{2}\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)|u|^2\mathrm{d}x$$

$$-\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)\left|Qz(\theta_t\omega)\right|^2\mathrm{d}x,\tag{40}$$

$$\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)h\left(u_t+\varepsilon t-Q(x)z(\theta_t\omega)\right)v\mathrm{d}x\leqslant\frac{\alpha}{4}\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)|v|^2\mathrm{d}x+\frac{1}{\alpha}\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)|z|^2\mathrm{d}x,\tag{41}$$

$$(1+\varepsilon)\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)v\cdot Qz\mathrm{d}x\leqslant\frac{1+\varepsilon}{2\alpha}\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)\left(Q(x)z(\theta_{r-\tau}\omega)\right)^2\mathrm{d}x$$

$$+\frac{\alpha(1+\varepsilon)}{2}\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)|v|^2\mathrm{d}x,\tag{42}$$

$$\alpha\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)z(\theta_t\omega)\Delta Q(x)\cdot v\mathrm{d}x\leqslant\frac{\alpha}{2}\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)\left|z(\theta_t\omega)\cdot Q\right|^2\mathrm{d}x+\frac{\alpha}{2}\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)|v|^2\mathrm{d}x,\tag{43}$$

$$\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)f(x,u)v\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)F(x,u)\mathrm{d}x+\varepsilon\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)f(x,u)u\mathrm{d}x$$

$$-\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)f(x,u)Q(x)z(\theta_t\omega)\mathrm{d}x,\tag{44}$$

由式(37)～(44)可得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)\left(|v|^2+(1-\alpha\varepsilon)|\nabla u|^2+(\xi+\varepsilon^2)|u|^2+2F(x,u)\right)\mathrm{d}x+\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)\frac{3\alpha+\varepsilon}{2}|v|^2\mathrm{d}x$$

$$\leqslant\frac{\sqrt{2}\,c(\alpha\varepsilon-1)}{r}\left(\|\nabla u\|^2+\|v\|^2\right)+c\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)\left(1+\left|Qz(\theta_t\omega)\right|^2+\left|\nabla Qz(\theta_t\omega)\right|^2+\left|Qz(\theta_t\omega)\right|^{\gamma+1}\right)\mathrm{d}x$$

$$+\frac{1}{\alpha}\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)g(x,t)\mathrm{d}x。\tag{45}$$

令 $X=|v|^2+(\xi+\varepsilon^2)|u|^2+(1-\alpha\varepsilon)|\nabla u|^2$，则式(45)可化为

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)(X+2F)\mathrm{d}x+\varepsilon\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)(X+2F)\mathrm{d}x$$

$$\leqslant c\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)\left(1+\left|Qz\left(\theta_t\omega\right)\right|^2+\left|\nabla Qz\left(\theta_t\omega\right)\right|^2+\left|Qz\left(\theta_t\omega\right)\right|^{\gamma+1}+\left|g\left(x,t\right)\right|^2\right)\mathrm{d}x$$

$$+\frac{\sqrt{2}\,c}{r}\left(\|\nabla u\|^2+\|v\|^2\right)。\tag{46}$$

在式(46)两边同乘$e^{\alpha t}$,并在$[\tau-t,\tau]$上积分得

$$\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)\left(X\left(\tau,\tau-t,\omega,X_0\right)+2F\left(x,u\left(\tau,\tau-t,\omega,u_0\right)\right)\right)\mathrm{d}x$$

$$\leqslant e^{-\varepsilon t}\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)\left(X_0+2F\left(x,u_0\right)\right)\mathrm{d}x+\frac{\sqrt{2}\,c}{r}\int_{\tau-t}^{\tau}e^{\varepsilon(s-\tau)}\left(\left\|\nabla u\left(s,\tau-t,\omega,u_0\right)\right\|^2+\left\|v\left(s,\tau-t,\omega,v_0\right)\right\|^2\right)\mathrm{d}s$$

$$+c\int_{\tau-t}^{\tau}e^{\varepsilon(s-\tau)}\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)\left(1+\left|Qz\left(\theta_t\omega\right)\right|^2+\left|\nabla Qz\left(\theta_t\omega\right)\right|^2+\left|Qz\left(\theta_t\omega\right)\right|^{\gamma+1}+\left|g\left(x,t\right)\right|^2\right)\mathrm{d}x\mathrm{d}s。\tag{47}$$

用$\theta_{-\tau}\omega$代替ω,式(47)可化为

$$\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)\left(X\left(\tau,\tau-t,\theta_{-\tau}\omega,X_0\right)+2F\left(x,u\left(\tau,\tau-t,\theta_{-\tau}\omega,u_0\right)\right)\right)\mathrm{d}x$$

$$\leqslant e^{-\varepsilon t}\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)\left(X_0+2F\left(x,u_0\right)\right)\mathrm{d}x+\frac{\sqrt{2}\,c}{r}\int_{\tau-t}^{\tau}e^{\varepsilon(s-\tau)}\left\|\nabla u\left(s,\tau-t,\theta_{-\tau}\omega,u_0\right)\right\|^2\mathrm{d}s$$

$$+\frac{\sqrt{2}\,c}{r}\int_{\tau-t}^{\tau}e^{\varepsilon(s-\tau)}\left\|v\left(s,\tau-t,\theta_{-\tau}\omega,v_0\right)\right\|^2\mathrm{d}s。\tag{48}$$

由$\phi_0\left(\theta_{-t}\omega\right)\in B\left(\theta_{-t}\omega\right)$和式(32)知,存在$\tilde{T}_1=\tilde{T}_1(B,\varepsilon,\omega)>0$,使得对任意的$t>\tilde{T}_1$,有

$$e^{-\alpha t}\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)\left(X_0+2F\left(x,u_0\right)\right)\mathrm{d}x\leqslant\varepsilon。\tag{49}$$

由引理3.1知,存在$\tilde{T}_2=\tilde{T}_2(B,\varepsilon,\omega)>0,\tilde{R}_1=\tilde{R}_1(\varepsilon,\omega)>1$,对$t>\tilde{T}_2,r>\tilde{R}_1$,有

$$\frac{c}{r}\int_{\tau-t}^{\tau}e^{\alpha(s-\tau)}\left(\left\|\nabla u\left(s,\tau-t,\theta_{-\tau}\omega,u_0\right)\right\|^2+\left\|v\left(s,\tau-t,\theta_{-\tau}\omega,v_0\right)\right\|^2\right)\mathrm{d}s\leqslant\varepsilon。\tag{50}$$

应用引理2.1知,取$\varepsilon=\dfrac{\sigma}{2(\gamma+1)}$,则存在$\tilde{T}_3=\tilde{T}_3(\varepsilon,\omega)>0,\tilde{R}_2(\varepsilon,\omega)>1$,当$t>\tilde{T}_3,r>\tilde{R}_2$时,有

$$c\int_{-\infty}^{0}\int_{|x|\geqslant r}e^{\sigma s}\left(\left\|Qz\left(\theta_s\omega\right)\right\|^2+\left\|\nabla Qz\left(\theta_s\omega\right)\right\|^2+\left\|Qz\left(\theta_s\omega\right)\right\|_{H^1}^{\gamma+1}\right)\mathrm{d}x\,\mathrm{d}s\leqslant\varepsilon。\tag{51}$$

由假设(H3)可得,存在$\tilde{R}_3=\tilde{R}_3(\varepsilon,\tau)>1$,使得对任意的$r>\tilde{R}_3$,有

$$c\int_{-\infty}^{\tau}e^{\sigma s}\int_{|x|\geqslant r}\left|g\left(x,s\right)\right|^2\mathrm{d}x\mathrm{d}s\leqslant\varepsilon。\tag{52}$$

令$\tilde{T}=\max\left\{\tilde{T}_1,\tilde{T}_2,\tilde{T}_3\right\},\tilde{R}=\max\left\{\tilde{R}_1,\tilde{R}_2,\tilde{R}_3\right\}$,结合上式,对任意的$t>\tilde{T},r>\tilde{R}$,有

$$\int_{\mathbf{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)\left(X\left(\tau,\tau-t,\theta_{-\tau}\omega,X_0\right)+2F\left(x,u\left(\tau,\tau-t,\theta_{-\tau}\omega,u_0\right)\right)\right)\mathrm{d}x\leqslant c\varepsilon。$$

令$\hat{\rho}=1-\rho$,给定$r\geqslant1$,且令

$$\begin{cases}\hat{u}\left(t,\tau,\omega,\hat{u}_0\right)=\hat{\rho}\left(\dfrac{|x|^2}{r^2}\right)u\left(t,\tau,\omega,u_0\right)\\[2ex]\hat{v}\left(t,\tau,\omega,\hat{v}_0\right)=\hat{\rho}\left(\dfrac{|x|^2}{r^2}\right)v\left(t,\tau,\omega,v_0\right)\end{cases},\tag{53}$$

则$\hat{\phi}\left(t,\tau,\omega,\hat{\phi}_0\right)=\left(\hat{u}\left(t,\tau,\omega,\hat{u}_0\right),\hat{v}\left(t,\tau,\omega,\hat{v}_0\right)\right)^{\mathrm{T}}$是式(14)~(16)在有界域$Q_{2r}$上的解。

将式(14)、(15)与$\rho\left(\frac{|x|^2}{r^2}\right)$相乘，并结合式(53)，有

$$\begin{cases}\frac{\mathrm{d}\hat{u}}{\mathrm{d}t}=\hat{v}-\varepsilon\hat{u}+\hat{\rho}Q(x)z\left(\theta_t\omega\right)\\ \frac{\mathrm{d}\hat{v}}{\mathrm{d}t}=\alpha\Delta\hat{v}+(1-\alpha\varepsilon)\Delta\hat{u}+\varepsilon\hat{v}-\left(\xi+\varepsilon^2\right)\hat{u}+(1+\varepsilon)\hat{\rho}Qz+\alpha\hat{\rho}z\Delta Q-\hat{\rho}h-\hat{\rho}f+\hat{\rho}g-u\Delta\hat{\rho}-2\nabla\hat{\rho}\nabla u\end{cases}。\tag{54}$$

考虑Q_{2r}上的特征问题$-\Delta\hat{u}=\lambda\hat{u},\hat{u}|_{\partial Q_{2r}}=0$，则存在一族特征函数$\left\{e_i\right\}_{i\in N}$以及相应的特征值$\left\{\lambda_i\right\}_{i\in N}$满足$\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_i\leqslant\cdots,\lambda_i\to\infty(i\to\infty)$，且$\left\{e_i\right\}_{i\in N}$是$L^2\left(Q_{2r}\right)$上的一组标准正交基。给定$n$，令$X_n=span\left\{e_1,e_2,\cdots,e_n\right\},P_n:L^2\left(Q_{2r}\right)\to X_n$是投影算子。

引理3.4　假设(H1)~(H3)成立，令$\{B(\omega)\}\in D,\phi_0(\omega)\in B(\omega)$，则对任意的$\varepsilon>0,\hat{R}=\hat{R}(\varepsilon,\omega)\geqslant 1,\hat{T}=\hat{T}(B,\varepsilon,\omega)>0,N=N(\varepsilon,\omega)>0$，使得当$t\geqslant\hat{T},r\geqslant\hat{R},n\geqslant N$时，有

$$\left\|\left(I-P_n\right)\hat{\phi}\left(t,\theta_{-t}\omega,\phi_0\left(\theta_{-t}\omega\right)\right)\right\|_{E\left(H_{2r}\right)}^2\leqslant\varepsilon。\tag{55}$$

证明　令$\hat{u}_{n,1}=P_n\hat{u},\hat{u}_{n,2}=\left(I-P_n\right)\hat{u},\hat{v}_{n,1}=P_n\hat{v},\hat{v}_{n,2}=\left(I-P_n\right)\hat{v}$，由式(55)可得

$$\hat{v}_{n,2}=\frac{\mathrm{d}\hat{u}_{n,2}}{\mathrm{d}t}+\varepsilon\hat{u}_{n,2}-Q(x)z\left(\theta_t\omega\right)。$$

在式(54)两边作用算子$I-P_n$，再与$\hat{v}_{n,2}$在$L^2\left(Q_{2r}\right)$上作内积得

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\hat{v}_{n,2}\right\|^2=&\ \varepsilon\left\|\hat{v}_{n,2}\right\|+\alpha\left\|\nabla\hat{v}_{n,2}\right\|+(1-\alpha\varepsilon)\left(\Delta\hat{u}_{n,2},\hat{v}_{n,2}\right)-\left(\xi+\varepsilon^2\right)\left(\hat{u}_{n,2},\hat{v}_{n,2}\right)-\left(I-P_n\right)\hat{\rho}\left(h,\hat{v}_{n,2}\right)\\&-\left(\hat{\rho}f,\hat{v}_{n,2}\right)+\left(\hat{\rho}g,\hat{v}_{n,2}\right)+\left(\rho(1-\varepsilon)Qz+\rho\alpha z\Delta Q,\hat{v}_{n,2}\right)-\left(u\Delta\rho+2\nabla\rho\nabla u,\hat{v}_{n,2}\right)。\end{aligned}\tag{56}$$

对式(56)的各项进行估计，

$$\begin{aligned}\left(\hat{u}_{n,2},\hat{v}_{n,2}\right)&=\left(\hat{u}_{n,2},\frac{\mathrm{d}\hat{u}_{n,2}}{\mathrm{d}t}+\varepsilon\hat{u}_{n,2}-Q(x)z\left(\theta_t\omega\right)\right)\\&\geqslant\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\hat{u}_{n,2}\right\|^2+\frac{\varepsilon}{4}\left\|\hat{u}_{n,2}\right\|^2-\frac{1}{\varepsilon}Q(x)\left\|z\left(\theta_t\omega\right)\right\|^2\cdot\left\|\hat{u}_{n,2}\right\|^2,\end{aligned}\tag{57}$$

$$\begin{aligned}\left(\Delta\hat{u}_{n,2},\hat{v}_{n,2}\right)&=-\left(\nabla\hat{u}_{n,2},\nabla\left(\frac{\mathrm{d}\hat{u}_{n,2}}{\mathrm{d}t}+\varepsilon\hat{u}_{n,2}-Q(x)z\left(\theta_t\omega\right)\right)\right)\\&\leqslant-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\hat{u}_{n,2}\right\|^2-\varepsilon\left\|\nabla\hat{u}_{n,2}\right\|^2+\nabla Q(x)\left|\nabla\hat{u}_{n,2}\right|\left|z\left(\theta_t\omega\right)\right|。\end{aligned}\tag{58}$$

由Young不等式可推得

$$\left(\alpha\hat{\rho}z\Delta Q(x),\hat{v}_{n,2}\right)\leqslant\frac{\alpha}{2}\left\|\left(I-P_n\right)z\left(\theta_t\omega\right)Q(x)\right\|^2+\frac{\alpha}{2}\left\|\hat{v}_{n,2}\right\|^2,$$

$$\begin{aligned}\left(\hat{\rho}f(x,u),\hat{v}_{n,2}\right)=&\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{\rho}f(x,u),\hat{u}_{n,2}\right)-\left(\hat{\rho}fu_t,\hat{u}_{n,2}\right)+\varepsilon\left(\hat{\rho}f,\hat{u}_{n,2}\right)-\left(\hat{\rho}f,\left(I-P_n\right)Qz\right)\\\geqslant&\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{\rho}f(x,u),\hat{u}_{n,2}\right)-\frac{\alpha}{4}\left\|\nabla\hat{u}_{n,2}\right\|^2-\lambda_{n+1}^{-\frac{1}{2}}\left\|u_t\right\|^2-\lambda_{n+1}^{\frac{\gamma-3}{2}}\left\|u_t\right\|^2\|u\|_{H^1}^{2\gamma-2}+\varepsilon\left(\hat{\rho}f,\hat{u}_{n,2}\right)\\&-\|u\|_{H^1}^{\gamma}\left\|\left(I-P_n\right)\hat{\rho}Qz\left(\theta_t\omega\right)\right\|-\left\|\left(I-P_n\right)\hat{\rho}Qz\left(\theta_t\omega\right)\right\|,\end{aligned}\tag{59}$$

$$\left(I-P_n\right)\hat{\rho}\left(h\left(v-\varepsilon u+Qz\right)\right)\geqslant\beta_1\left\|\hat{v}_{n,2}\right\|^2+\beta_1\left(\hat{\rho}z\left(\theta_t\omega\right)Q,\hat{v}_{n,2}\right)-\beta_2\varepsilon\left(\hat{u}_{n,2},\hat{v}_{n,2}\right),$$

$$\begin{aligned}\left(\hat{\rho}g(x,t),\hat{v}_{n,2}\right)&\leqslant\left\|\left(I-P_n\right)\hat{\rho}g(x,t)\right\|^2+\frac{\beta_1-\varepsilon}{8}\left\|\hat{v}_{n,2}\right\|^2-\left(u\Delta\hat{\rho}+2\nabla\rho\nabla u,\hat{v}_{n,2}\right)\\&\leqslant\frac{c}{r^4}\|u\|^2+\frac{c}{r^2}\|\nabla u\|^2+\frac{\beta_1-\varepsilon}{4}\left\|\hat{v}_{n,2}\right\|^2。\end{aligned}\tag{60}$$

由式(56)~(60)可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\hat{\phi}_{n,2}\right\|_{E\left(Q_{2r}\right)}^{2}+2\left(\hat{\rho}f\left(x,u\right),\hat{u}_{n,2}\right)\right)+\varepsilon\left(\left\|\hat{\phi}_{n,2}\right\|_{E\left(Q_{2r}\right)}^{2}+2\left(\hat{\rho}f\left(x,u\right),\hat{u}_{n,2}\right)\right)$$
$$\leqslant\frac{3\alpha}{4}\left\|\nabla\hat{u}_{n,2}\right\|^{2}+\left|\nabla Q\left(x\right)\right\|\nabla\hat{u}_{n,2}\right|\cdot|z|-\frac{1}{2\varepsilon}Q\left(x\right)\|z\|^{2}\left\|\hat{u}_{n,2}\right\|^{2}+\frac{c}{r^{4}}\|u\|^{2}+\frac{c}{r^{2}}\|\nabla u\|^{2}$$
$$+\lambda_{n+1}^{-\frac{1}{2}}\left\|u_{t}\right\|^{2}+\lambda_{n+1}^{\frac{\gamma-3}{2}}\left\|u_{t}\right\|^{2}\|u\|_{H^{1}}^{2\gamma-2}+\|u\|_{H^{1}}^{\gamma}\left\|\left(I-P_{n}\right)\hat{\rho}Qz\right\|+\left\|\left(I-P_{n}\right)\hat{\rho}zQ\right\|+\left\|\left(I-P_{n}\right)\hat{\rho}g\right\|^{2}。\tag{61}$$

由$1\leqslant\gamma\leqslant3,\lambda_{n}\rightarrow\infty$可知,存在$\hat{N}_{1}=\hat{N}_{1}(\varepsilon)>0,\hat{R}_{1}=\hat{R}_{1}(\varepsilon)>0$,使得对任意的$n\geqslant\hat{N}_{1},r\geqslant\hat{R}_{1}$,有

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\hat{\phi}_{n,2}\right\|_{E\left(Q_{2r}\right)}^{2}+2\left(\hat{\rho}f\left(x,u\right),\hat{u}_{n,2}\right)\right)+\varepsilon\left(\left\|\hat{\phi}_{n,2}\right\|_{E\left(Q_{2r}\right)}^{2}+2\left(\hat{\rho}f\left(x,u\right),\hat{u}_{n,2}\right)\right)$$
$$\leqslant\frac{c}{r^{4}}\|u\|^{2}+\frac{c}{r^{2}}\|\nabla u\|^{2}+c\left\|\left(I-P_{n}\right)\hat{\rho}g\left(x,t\right)\right\|^{2}+c\|u\|_{H_{1}}^{\gamma}\left\|\left(I-P_{n}\right)\hat{\rho}Q\left(x\right)z\left(\theta_{t}\omega\right)\right\|$$
$$+c\varepsilon\left(1+\left|z\left(\theta_{t}\omega\right)Q\left(x\right)\right|^{2}+\left\|u_{t}\right\|^{6}+\|u\|_{H^{1}}^{6}\right)。\tag{62}$$

在式(62)两边同乘$e^{\alpha t}$,并在$[\tau-t,\tau]$上积分,对任意的$n\geqslant\hat{N}_{1},r\geqslant\hat{R}_{1}$,有

$$\left\|\hat{\phi}_{n,2}\left(\tau,\tau-t,\omega,\hat{\phi}_{n,2,0}\right)\right\|_{E\left(Q_{2r}\right)}^{2}+2\left(\hat{\rho}f\left(x,u\left(\tau,\tau-t,\omega,u_{0}\right)\right),\hat{u}_{n,2}\left(\tau,\tau-t,\omega,\hat{u}_{n,2,0}\right)\right)$$
$$\leqslant e^{-\alpha t}\left(\left\|\hat{\phi}_{n,2,0}\right\|_{E\left(Q_{2r}\right)}^{2}+2\left(\hat{\rho}f\left(x,u_{0}\right),\hat{u}_{n,2,0}\right)\right)+c\int_{\tau-t}^{\tau}e^{\alpha(s-\tau)}\left\|\left(I-P_{n}\right)\hat{\rho}g\left(x,s\right)\right\|^{2}\mathrm{d}s$$
$$+\frac{c}{r^{4}}\int_{\tau-t}^{\tau}e^{\alpha(s-\tau)}\left\|u\left(s,\tau-t,\omega,u_{0}\right)\right\|^{2}\mathrm{d}s+\frac{c}{r^{2}}\int_{\tau-t}^{\tau}e^{\alpha(s-\tau)}\left\|\nabla u\left(s,\tau-t,\omega,u_{0}\right)\right\|^{2}\mathrm{d}s$$
$$+c\varepsilon\int_{\tau-t}^{\tau}e^{\alpha(s-\tau)}\left(1+\left|z\left(\theta_{t}\omega\right)Q\left(x\right)\right|^{2}+\left\|u_{t}\right\|^{6}+\|u\|_{H^{1}}^{6}\right)\mathrm{d}s。\tag{63}$$

用$\theta_{-\tau}\omega$代替ω,则式(63)可化为

$$\left\|\hat{\phi}_{n,2}\left(\tau,\tau-t,\theta_{-\tau}\omega,\hat{\phi}_{n,2,0}\right)\right\|_{E\left(Q_{2r}\right)}^{2}+2\left(\hat{\rho}f\left(x,u\left(\tau,\tau-t,\theta_{-\tau}\omega,u_{0}\right)\right),\hat{u}_{n,2}\left(\tau,\tau-t,\theta_{-\tau}\omega,\hat{u}_{n,2,0}\right)\right)$$
$$\leqslant e^{-\alpha t}\left(\left\|\hat{\phi}_{n,2,0}\right\|_{E\left(Q_{2r}\right)}^{2}+2\left(\hat{\rho}f\left(x,u_{0}\right),\hat{u}_{n,2,0}\right)\right)+c\int_{-\infty}^{\tau}e^{\alpha(s-\tau)}\left\|\left(I-P_{n}\right)\hat{\rho}g\left(x,s\right)\right\|^{2}\mathrm{d}s$$
$$+\frac{c}{r^{4}}\int_{\tau-t}^{\tau}e^{\alpha(s-\tau)}\left\|u\left(s,\tau-t,\theta_{-\tau}\omega,u_{0}\right)\right\|^{2}\mathrm{d}s+\frac{c}{r^{2}}\int_{\tau-t}^{\tau}e^{\alpha(s-\tau)}\left\|\nabla u\left(s,\tau-t,\theta_{-\tau}\omega,u_{0}\right)\right\|^{2}\mathrm{d}s$$
$$+c\varepsilon\int_{\tau-t}^{\tau}e^{\alpha(s-\tau)}\left(1+\left|z\left(\theta_{-t}\omega\right)Q\left(x\right)\right|+\left\|u_{t}\left(s,\tau-t,\theta_{-t}\omega,u_{0}\right)\right\|^{6}+\left\|u\left(s,\tau-t,\theta_{-t}\omega,u_{0}\right)\right\|_{H^{1}}^{6}\right)\mathrm{d}s。\tag{64}$$

应用(H1),且由$\phi_{0}\in\tilde{D}\left(\tau-t,\theta_{-\tau}\omega\right)\in D$可知,当$\hat{T}_{1}=\hat{T}_{1}(\tau,\varepsilon,D,\omega)>0,\hat{R}_{1}=\hat{R}_{1}(\tau,\varepsilon,\omega)>1$,使得对任意的$t>\hat{T}_{1},r>\hat{R}_{1}$,有

$$e^{-\alpha t}\left(\left\|\hat{\phi}_{n,2,0}\right\|_{E\left(Q_{2r}\right)}^{2}+2\left(\hat{\rho}f\left(x,u_{0}\right),\hat{u}_{n,2,0}\right)\right)\leqslant\varepsilon。\tag{65}$$

由假设(H3),存在$\hat{N}=\hat{N}(\tau,\omega,\varepsilon)>0$,使得对任意的$n>\hat{N}$,有

$$c\int_{-\infty}^{\tau}e^{\alpha(s-\tau)}\left\|\left(I-P_{n}\right)\hat{\rho}g\left(x,s\right)\right\|^{2}\mathrm{d}s\leqslant\varepsilon。\tag{66}$$

由引理2.1知,存在$\hat{T}_{2}=\hat{T}_{2}(\tau,\varepsilon,D,\omega)>0,\hat{R}_{2}=\hat{R}_{2}(\tau,\varepsilon,\omega)>1$,使得对任意的$t>\hat{T}_{2},r>\hat{R}_{2}$,有

$$\frac{c}{r^{4}}\int_{\tau-t}^{\tau}e^{\alpha(s-\tau)}\left\|u\left(s,\tau-t,\theta_{-\tau}\omega,u_{0}\right)\right\|^{2}\mathrm{d}s+\frac{c}{r^{2}}\int_{\tau-t}^{\tau}e^{\alpha(s-\tau)}\left\|\nabla u\left(s,\tau-t,\theta_{-\tau}\omega,u_{0}\right)\right\|^{2}\mathrm{d}s\leqslant\varepsilon。\tag{67}$$

由引理2.1和引理3.1知,存在$\hat{T}_{3}=\hat{T}_{3}(\tau,\varepsilon,D,\omega)>0$,使得对任意的$t>\hat{T}_{3}$,

$$\int_{\tau-t}^{\tau}e^{\alpha(s-\tau)}\left(\left|z\left(\theta_{-t}\omega\right)\right|^{2}+\left\|u_{t}\left(s,\tau-t,\theta_{-t}\omega,u_{0}\right)\right\|^{6}+\left\|u\left(s,\tau-t,\theta_{-t}\omega,u_{0}\right)\right\|_{H^{1}}^{6}\right)\mathrm{d}s<\infty。\tag{68}$$

令 $\hat{T}=\max\left\{\hat{T}_1,\hat{T}_2,\hat{T}_3\right\}$，$\hat{R}=\max\left\{\hat{R}_1,\hat{R}_2\right\}$，结合式（63）~（67）可得，对任意的 $t>\hat{T}$，$r>\hat{R}$，$n>\hat{N}$，有

$$\left\|\hat{\phi}_{n,2}\left(\tau,\tau-t,\theta_{-t}\omega,\hat{\phi}_{n,2,0}\right)\right\|^2_{E\left(Q_{2r}\right)}\leqslant c\varepsilon。$$

由引理3.1 ~ 3.4，我们有以下主要结果：

引理3.5 设 $Q(x)\in H^1\left(\mathbf{R}^n\right)$，假设（H1）~（H3）成立，令 $\left\{B(\omega)\right\}\in D$ 且 $\phi_0(\omega)\in B(\omega)$，则由式（1）~（2）产生的随机动力系统 Φ 在 $E\left(\mathbf{R}^n\right)$ 上存在唯一的 D -拉回随机吸引子

$$A=\left\{A(\tau,\omega):\tau\in R,\omega\in\Omega\right\}\in D。$$

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